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INVESTIGATION OF THE $K^{39}(d, p)K^{40}$ AND $Ca^{40}(d, p)Ca^{41}$ REACTIONS

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Nuclear emulsions were used to study the angular distributions of three groups of protons from the $K^{39}(d, p)K^{40}$ reaction for a deuteron energy of 4 Mev, and also the angular distribution of the long-range proton group from the $Ca^{40}(d, p)Ca^{41}$ reaction with 1.3, 2.2, and 4.0 Mev deuterons. The measured angular distributions do not agree with those calculated from the Butler theory. Comparison of the results obtained for the two reactions enables us to draw some conclusions concerning the low-lying states of the K^{40} nucleus.

THE (d, p) stripping reaction has been investigated for the principal isotopes of all elements with $Z \leq 20$. The only exception is the $K^{39}(d, p)K^{40}$ reaction; no data have been published concerning the angular distribution of the protons formed in this reaction.

In 1954 one of the authors[1] studied this reaction for the purpose of assigning quantum numbers to the low-lying states of K^{40}. However it appeared that the angular distributions could not be interpreted uniquely using the Butler theory.[2] It was necessary to assume configuration mixing or to suppose that the Butler theory is not applicable to this case. The latter appeared quite probable, since the Butler theory does not take account of the Coulomb or the nuclear interaction between the particles participating in the reaction. At the same time, such interactions are important in this case because the energy of the bombarding deuterons (4 Mev) is below the Coulomb barrier (5.65 Mev).

In order to find the cause of the discrepancy between the experimental results and the theory, we first made some additional studies of the $K^{39}(d, p)K^{40}$ reaction, and also measured the angular distribution of the protons which are produced in the $Ca^{40}(d, p)Ca^{41}$ reaction when the Ca^{41} nucleus is formed in the ground state. The latter reaction was selected because there are the same number of neutrons in the K^{39} and Ca^{40} nuclei and their charges are almost the same. In addition, the calcium reaction had already been studied for a deuteron energy of about 8 Mev (Ref. 3) and good agreement with the Butler theory had been obtained in this case.

In order to understand the changes in shape of the angular distribution with changing deuteron energy, the $Ca^{40}(d, p)Ca^{41}$ reaction was also studied for deuteron energies of 2.2 and 1.3 Mev.

EXPERIMENTAL METHOD

The experiments were carried out at the 72 cm cyclotron of Moscow State University. The required changes in deuteron energy were achieved by changing the mode of operation of the cyclotron. Consequenly the beam intensity was practically unchanged when the deuteron energy was lowered. The extracted, focused beam of particles passed through a collimator into a special cylindrical chamber, 26 cm in diameter, placed at a distance of about 5.5 meters from the accelerator. The energy spread of the deuterons traversing the collimator did not exceed ± 40 kev. A thin target was placed at the center of the chamber, with nuclear plates at various angles around it for recording the reaction products. Aluminum or silver absorbers were placed in front of the plates to cut out elastically scattered deuterons and long-range protons.

Thin targets containing potassium were prepared by evaporation of potassium iodide onto a gold foil backing, followed by admission of iodine vapor into the vacuum system.[4] Calcium targets were obtained by evaporating metallic calcium. Immediately after preparation, such a target changes its color as the calcium oxidizes, but then over a long period no noticeable changes occur in the target. The amount of material deposited on the backing was determined by weighing the target after irradiation. In this way the thickness of the layer was measured to an accuracy of $\pm 0.05 - 0.08$ mg/cm^2 (for a total layer thickness of $0.2 - 0.9$ mg/cm^2).

The current integrator which was used to measure the deuteron beam current was calibrated in terms of the elastic scattering of deuterons on gold, where it was assumed that the scattering is pure Rutherford scattering. The accuracy of this calibration, including errors in measurement of the thickness of the gold target and the energy of the bombarding particles, does not exceed 12 – 15%. Consequently the possible overall uncertainty in determining the absolute cross section of the reaction is 30 – 40%. The accuracy of the determination of relative cross sections for the $Ca^{40}(d, p)Ca^{41}$ reaction with different energies of the deuterons is considerably higher, since the same target was used in all the experiments. In addition, after carrying out the main irradiations at 4, 2.2, and 1.3 Mev, a check run was made at 4 Mev. The ratio of the differential cross sections measured at several angles in the check run to the cross sections at the same angles in the main run was 0.99, which showed that the calcium content of the target was practically unchanged.

In studying the reaction products from bombardment with moderate energy deuterons of any but the very lightest nuclei, the problem of impurities in the target is important. Light element impurities are especially harmful, since their reactions occur with a considerable probability even for low bombarding energies. Since purified reagents were used for the targets, the only substances which can appear in significant amounts in the targets are oxygen, carbon, and nitrogen, which are hard to get rid of. However, only those protons which come from the $C^{13}(d, p)C^{14}$ and $N^{14}(d, p)N^{15}$ reactions, in which the final nucleus is formed in its ground state, have sufficient energy to pass through the absorbers placed in front of the plates. The small number of proton tracks from the nitrogen reaction, which were observed in the measurements with 2.2 and 1.3 Mev deuterons, did not interfere with the measurements since their length is much greater than that of the true tracks. The protons from $C^{13}(d, p)C^{14}$ have an energy close to that of the particles formed in the potassium and calcium reactions, and can therefore be a serious contamination. Experiment showed that the proton group from the reaction on C^{13} is noticeable only for the $Ca^{40}(d, p)Ca^{41}$ reaction with 1.3 Mev deuterons. Using data from the literature on the $C^{13}(d, p)C^{14}$ cross section,[5] we can compute that there is little carbon in the calcium target, and that the target consists mainly of calcium oxide (approximately 9 CaO molecules per $CaCO_3$ molecule). At angles of emergence below 50° we were unable to resolve the proton groups from C^{13} and Ca^{40}.

TABLE I. Excitation Energy (Mev) of Low-Lying Levels of K^{40}

Reference 6	Reference 4	Reference 7	Our Results	Proton Group
0 0.03	0	0	0	p_{01}
0.80 0.89	0.81	0.76	0.82	p_{23}
	2.01	2.06		
		2.28	2.08	p_4
		2.40		
	2.56	2.56		
		2.77		

NIKFI Ia-2 plates with an emulsion thickness of 100 μ were used in the experiments. The number of plates used in the various experiments ranged from 21 to 32. In order to speed up the search, in most cases the energy spectrum was taken for each plate with low statistics. Then the limits of a particular proton group were found from the histogram, and the number of tracks belonging to this group was counted in a definite area of the plate.

The accuracy of the angular distributions is determined, aside from statistical fluctuations, by many other factors (such as the accuracy in alignment of plates, target, collimator, etc.) which are difficult to estimate. However, the comparison of the angular distribution of elastic scattering of deuterons on gold with the Rutherford formula, and the comparison of the results obtained in the various experiments for plates placed at the same angle to right and left of the beam, show that the rms spread in the individual points does not exceed 8 – 10% (while the statistical errors are 5 – 6%). We may therefore assume that the errors in the measurements of angular distributions, which are not due to statistical fluctuations, are surely less than 10%.

EXPERIMENTAL RESULTS

The $K^{39}(d, p)K^{40}$ reaction was studied for a deuteron energy of 4 Mev. Since the normal mixture of potassium isotopes was used in the present work, the only proton group which could be selected with certainty is that from the most abundant isotope K^{39} (93.3%). Table I gives the available data on the lowest excited levels of K^{40}, as obtained by Sailor[4] using proton absorption in Al, by Buechner et al.[6] from magnetic analysis of the protons, and by Adiasevich, Groshev, et al.[7] from the spectrum of γ rays emitted by K after thermal neutron absorption. Figure 1 shows the energy spectrum which we found for protons emitted at an angle of 41° with respect to the incident beam. Three proton groups are clearly visible on the histrogram. As was to be expected, we could not resolve close levels with the plates, so that the group p_{01} contains protons which are emitted when K^{40} is produced in its ground and first excited states, and the group p_{23} contains protons corresponding to the levels at 0.80 and 0.89 Mev. The group p_4 is also apparently an unresolved doublet or triplet.

The angular distributions obtained for the proton groups p_{01}, p_{23}, and p_4 are shown in Figs. 2, 3, and 4 respectively. The abscissa is the angle of emergence of the proton in the c.m. system, and the ordinate is the differential cross section in mbarn/sterad. The statistical errors are shown on the figures. The angular distributions for the proton groups p_{01} and p_{23} were found from three independent experiments carried out at different times. The results coincided within the limits of error. Figures 2 and 3 show the angular distributions for the series in which the plates were set at a large number of angles.

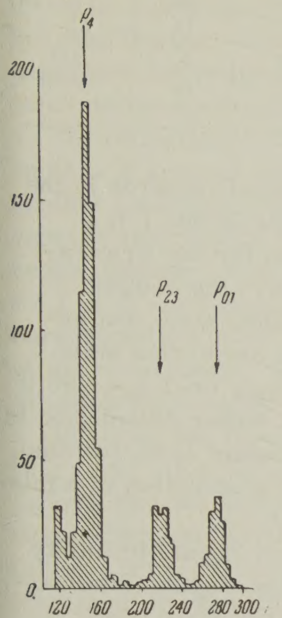

FIG. 1. Spectrum of protons from $K^{39}(d, p)K^{40}$. The abscissa is the number n of divisions on the microscope ocular scale (1 division = 1.41 μ).

The angular distribution of the proton group p_0, which corresponds to formation of Ca^{41} in its ground state in the $Ca^{40}(d, p)Ca^{41}$ reaction, was measured for deuteron energies of 4.0, 2.2, and 1.3 Mev. The experimental results are shown in Figs. 5 — 7. For 1.3 Mev deuterons, the angular distribution could only be measured with very poor statistics, and as already mentioned, only in the angular range above 50°.

Table II gives the values of the total cross section for the various proton groups. In the case of the groups p_{01} and p_{23} from the $K^{39}(d, p)K^{40}$ reaction, the averages of the results of the three sets of experiments are given; the rms fluctuation of individual measurements is 30%.

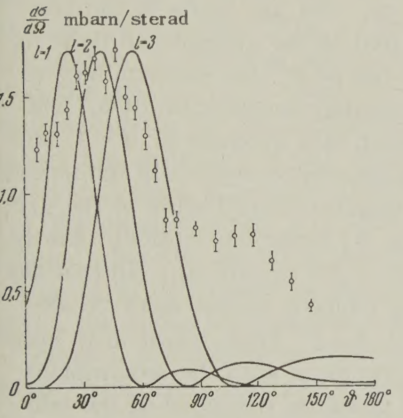

FIG. 2. Angular distribution of the group p_{01} from K. The solid curves are computed from the Butler formula.

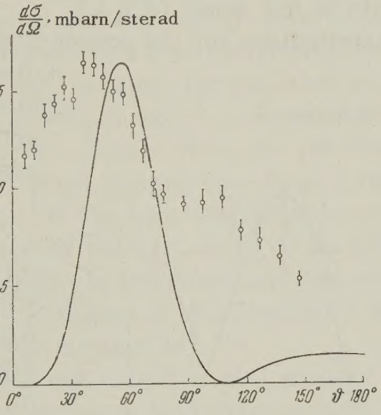

FIG. 3. Angular distribution of the group p_{23} from K. The solid curve is computed from the Butler formula with $\ell = 3$.

DISCUSSION OF RESULTS

In Fig. 2, which shows the angular distribution of the proton group p_{01} from the $K^{39}(d, p)K^{40}$ reaction, we give the curves calculated from the Butler[2] formula for orbital angular momentum of the captured neutron equal to $\ell = 1, 2, 3$ (the nuclear radius is set equal to 5.8 $\times 10^{-13}$ cm in the computations). Comparison of the shapes of these curves with the measured angular distribution shows that even moderately satisfactory agreement can be obtained only by assuming either that $\ell = 2$ or that the angular distribution is determined by the sum of $\ell = 1$ and $\ell = 3$ (with relative contributions of 1 and 8 respectively). The first assumption contradicts the single-particle model, ac-

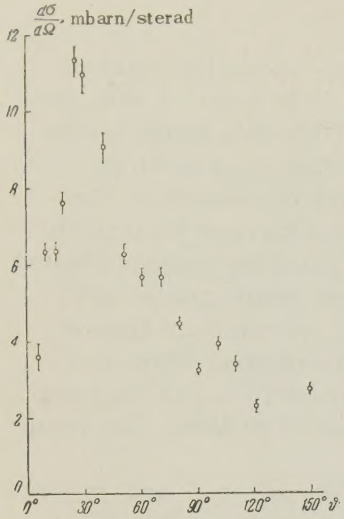

FIG. 4

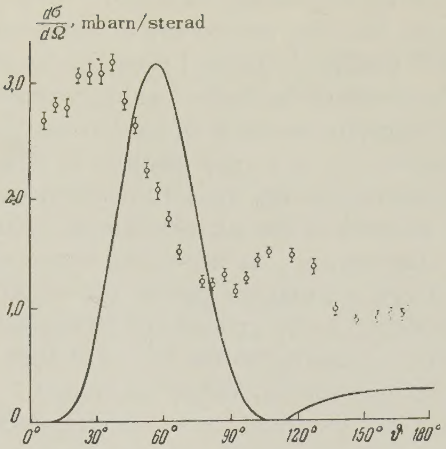

FIG. 5

FIG. 4. Angular distribution of group p_4 from potassium.

FIG. 5. Angular distribution of group p_0 from calcium for 4.0 Mev deuterons. The solid curve is computed from the Butler formula with $\ell = 3$.

TABLE II. Total Cross Sections

Reaction	Deuteron Energy, Mev	Proton Group	σ, mbarn
$K^{39}(d, p)K^{40}$	4.0	p_{01}	12
	4.0	p_{23}	13
	4.0	p_4	58
$Ca^{40}(d, p)Ca^{41}$	4.0	p_0	21
	2.2	p_0	2.2
	1.3	p_0	0.04

cording to which the odd neutron in the K^{40} nucleus should be in the 1 $f_{7/2}$ state. It is true that for the first excited level (0.03 Mev) the neutron might be in some other state, but according to the shell model this state should be either 2 $p_{3/2}$ or 1 $f_{5/2}$. So if we assume that the Butler calculation is applicable to the present case, the shell model requires the assumption of a mixture of $\ell = 1$ and $\ell = 3$.

However, the very great similarity between the results obtained in the present work, at a deuteron energy of 4.0 Mev for the angular distributions of the proton groups p_{01} from the $K^{39}(d, p)K^{40}$ reaction (Fig. 2) and p_0 from $Ca^{40}(d, p)Ca^{41}$ (Fig. 5), indicate that the orbital angular momentum of the neutron captured by potassium is 3. On the one hand the angular distribution of the group p_0 from the $Ca^{40}(d, p)Ca^{41}$ reaction, as measured by Holt and Marsham,[3] agrees with the calculated curve for $\ell = 3$. Since they used deuterons with an energy of 8 Mev, which is above the top of the Coulomb barrier, there is no reason in their case for doubting that the Butler theory is applicable. On the other hand, the neutron number is 20 in both K^{39} and Ca^{40}, so that the neutron states in the final nuclei should be the same (1 $f_{7/2}$).

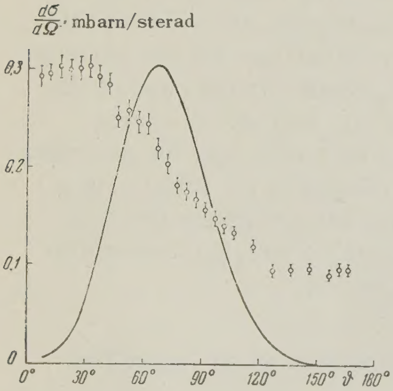

FIG. 6

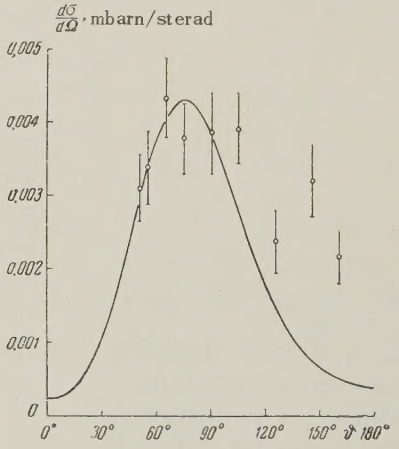

FIG. 7

FIG. 6. Angular distribution of group p_0 from calcium for 2.2 Mev deuterons. The solid curve is computed from the Butler formula with $\ell = 3$.

FIG. 7. Angular distribution of group p_0 from calcium for 1.3 Mev deuterons. The solid curve is computed from the Butler formula with $\ell = 3$.

Thus the comparison of the angular distributions for the proton groups p_{01} and p_0 leads to the conclusion that in the ground and first excited states of K^{40} the neutron has an orbtial angular momentum of 3. The assumption of a mixture of $\ell = 1$ and $\ell = 3$ is therefore obviously incorrect.

The angular distribution of the group p_{23} (Fig. 3) coincides almost exactly with that of the group p_{01}, so that the captured neutron should here be assigned to $\ell = 3$. This result is in complete agreement with the assumption that the quantum numbers of the odd neutron are the same (1 $f_{7/2}$) in the four lowest states of K^{40}, and that the existence of these four states is caused by different relative orientations of the angular momenta of the proton and neutron. From the measured relative intensities of the respective proton groups,[6] it has been possible to make assignments of all four levels, giving 4^-, 3^-, 2^-, and 5^-, in order of increasing energy.[8] This is confirmed by our results, since the average of the intensity ratio for groups p_{23} and p_{01} for all angles was equal to unity (more precisely, 0.97). If we assume that the cross section is proportional to $(2I_f + 1)$, where I_f is the spin of the final state, the equality of the intensities requires that levels with spins of 4 and 3 contribute to the one group, and levels with spins 2 and 5 to the other.

We should mention that the differential cross sections for the group $p_0(Ca^{40})$ and the sum of the differential cross sections for groups p_{01} and p_{23} (K^{39}) are practically the same at the peak of the angular distribution. Since the statistical factors $(2I_f + 1)/(2I_i + 1)$ are the same in both cases, this indicates that the reduced widths of the final states are equal, which once again confirms the validity of our conclusions concerning the low-lying excited states of K^{40}.

The differential cross section for formation of group p_4 (Fig. 4), both at the maximum and at large angles, is several times greater than the cross section for formation of groups p_{01} and p_{23}. This is still another indication that this group corresponds to several levels in the K^{40} nucleus. The maximum in the angular distribution of this group is considerably sharper, and is at a smaller angle, than that of the longer range protons. It is therefore reasonable to assume that the angular distribution is mainly determined by $\ell = 1$ or 2, and that $\ell = 1$ is more probable, since the large differential cross section is evidence that the corresponding level is a single-particle level and according to the shell model d-states should not appear at low excitations. So apparently among the levels corresponding to the group p_4 there are one or more in which the neutron is in a $2p_{3/2}$ state.

By using the results for the $Ca^{40}(d, p)Ca^{41}$ reaction with 4.0 Mev deuterons, we have succeeded in getting some information about the low-lying excited states of K^{40}. In addition, from these same results it follows that the Butler[2] theory, as usually stated, is not applicable to the description of angular distributions of protons produced in reactions of 4 Mev deuterons with nuclei having Z around 20. The main difference between the experimental angular distributions and those calculated from this theory is that the maximum in the differential cross section, which is a characteristic feature of the stripping reaction, is shifted toward smaller angles and is broadened. This fact was already pointed out in our paper[9] on the $S^{32}(d, p)S^{33}$ reaction. The displacement and broadening of the maximum is considerably greater for calcium and potassium than for sulfur. This may be explained first of all by the difference in nuclear charge, and secondly by the difference in orbital angular momentum of the captured neutron ($\ell = 2$ for sulfur and $\ell = 3$ for calcium and potassium). The results found for the $Ca^{40}(d, p)Ca^{41}$ reaction with 2.2 Mev deuterons (Fig. 6) show that for this case the maximum of the differential cross section is close to 0°, i.e., the difference between the experimental and theoretical distributions increases with decreasing deuteron energy. At 1.3 Mev (Fig. 7) the cross section apparently becomes more isotropic; if there is a rise at low angles, it starts at angles below 50°. Despite the fact that its position is close to the calculated location, the maximum, of which there is a hint around 80 – 90°, can hardly be the principal maximum which is characteristic for the stripping process. In any case, the results found with 4.0 and 2.2 Mev deuterons make this unlikely. If this maximum is not caused by a chance fluctuation of the points because of the poor statistics, then it is more likely that it is the secondary maximum associated with the stripping process (cf. for example, Fig. 5), or is caused by the symmetry of the differential cross section about 90° which occurs when the reaction goes primarily via the compound nucleus. In the latter case, as shown by calculation of Pratt[10] based on the statistical theory of Wolfenstein,[11] if the final nucleus is formed in a $7/2^-$ state, the cross section has a maximum around 90°.

Since a considerable difference between experimental angular distributions and those calculated from Butler's formula appears at deuteron energies less than or equal to the height of the Coulomb barrier, and since this difference increases with increasing nuclear charge and decreasing deuteron energy, it is obvious that for a correct description of the experimental results we must include both the Coulomb interaction and the nuclear interaction between the particles participating in the reaction. The fact that the maximum is shifted toward 0° shows that the nuclear interaction is very important at low deuteron energies, since the Coulomb interaction by itself shifts the peak toward larger angles,[12] i.e., in the opposite direction.

In conclusion, it is a pleasure to express our thanks to S. S. Vasil'ev for his interest in the work, and to the operating crew of the cyclotron, in particular to G. V. Kosheliaev, A. A. Danilov, and V. P. Khlapov.

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DISSOCIATION OF C^{12} INTO THREE ALPHA PARTICLES INDUCED BY FAST NEUTRONS

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The energy dependence of the effective cross-section for the neutron-induced disintegration of C^{12} into three α particles was investigated for neutron energies between threshold and 19 Mev. Information about the mechanism by which this reaction goes (in the indicated energy range) was obtained from calculations of the excitation energy of intermediate nuclei formed during the decay, and also from the distributions of the reaction products in space and energy.

MANY authors have studied the disintegration of C^{12} into three α particles as induced by fast neutrons. The most detailed examination of this reaction was carried out by Frye, Rosen, and Stewart,[1] whose work extends the data given by earlier workers (Ref. 2, and others). However, several points remained unclear, in particular the energy dependence of the effective cross section near the threshold and the maximum, how the mechanism of the reaction depended on the energy of the incident neutrons, and the question of whether direct interaction processes occurred at neutron energies of 18 to 20 Mev.

In this paper we investigate the energy dependence of the effective cross section for the decay of C^{12} into three α particles from threshold ($Q = -7.28$ Mev) to 19 Mev, and try to see how the mechanism of the decay depends on the energy of the incident neutrons.

Decay stars were observed both in NIKFI-Ia-2 emulsions and also in specially prepared layered emulsions loaded with a powder of spectroscopically pure carbon, the grain size being about 1 μ. The carbon powder enabled us to distinguish between three-pronged stars corresponding to the decay $C^{12} \rightarrow 3\alpha$ and three-pronged stars due to the reaction $N^{14}(n, 2\alpha)Li^7$, and also enabled us to take into account decays which looked like two-pronged stars because the third α particle had too little energy to make a track in the emulsion, thus giving a better value for the effective cross section.

The plates were exposed to neutrons from a thick lithium target bombarded by deuterons accelerated to 4 Mev in a cyclotron. Before hitting the emulsion, which was inclined at an angle of 6° to the incident current, the neutrons passed through a system of collimators. The cassettes containing the emulsions were shielded from scattered neutrons.

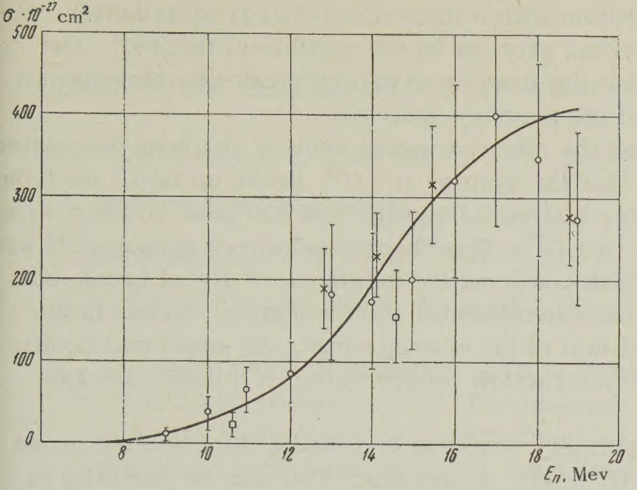

FIG. 1. The effective cross section for the decay of C^{12} into three α particles when bombarded by fast neutrons. ○ — data of the work reported in this paper, × — data from Ref. 1, □ — data from Ref. 2. Solid curve — the cross section according to Sachs' semi-empirical formula,[4] with $R = 0.88$, $a = 1.06$.

More than 500 stars, corresponding to the disintegration of C^{12} into three α particles, were observed. Taking the direction of the incident neutron to be known, the laws of energy and momentum conservation were used (see Ref. 2) to calculate, for each star, the energy of the incident neutron, the energy of the inelastically scattered neutron, and the direction in which the inelastically scattered neutron flew away. The neutron current was obtained from the recoil protons in the same plate used to observe the decay stars.

The wide energy range over which the lithium neutrons were distributed allows us to obtain the energy dependence of the cross section near threshold, and also to get a better value for the position of the maximum (Fig. 1). As one can see, our cross section agrees with the data of Refs. 1 and 2 within experimental error. Changes in the form of the cross section can be explained by the presence of various mechanisms for the decay, the probability associated with each mode depending on the energy of the incident neutrons.

The two fundamental reactions involved in the disintegration of C^{12} into three α particles are (n, n') and (n, α); each can go through various intermediate nuclei in varying states of excitation. The reaction $C^{12}(n, n')$ can go in the following ways: (a) simultaneous disintegration into a neutron and three α particles, $C^{12}(n, n'3\alpha)$; (b) the neutron is inelastically scattered from C^{12}, and the excited carbon nucleus then breaks up, all at once, into three particles, $C^{12}(n, n')C^{12*}3\alpha$; (c) the excited C^{12} nucleus can emit an α particle, and the residual Be^8 nucleus in its ground state or an excited one then breaks up into two α particles. The reaction (n, α) on the nucleus C^{12} can go in two ways: in the first way, the excited Be^9 nucleus emits a neutron, leaving Be^8 in either its ground state or an excited one, with subsequent decay into two α particles; in the second way, the Be^9 nucleus emits an α particle, leaving the unstable He^5 nucleus which then breaks up into an α particle and a neutron.

In order to clarify the mechanism by which the reaction goes at various energies of the primary neutrons, we calculated the excitation energies of the intermediate nuclei C^{12}, Be^9, and Be^8 for the observed stars. The excitation energies so calculated corresponded, more or less, to the known energy levels of these nuclei. This indicates that in a given energy range there is a definite probability for decay through a given intermediate nucleus; as the energy of the primary neutrons increases, higher and higher energy levels appear in the intermediate nuclei, while the relative probabilities for decay through these levels tend to decrease. The table shows, for each energy of the primary neutrons, the fraction of the total number of decays which go through various levels of the intermediate nuclei. From an examination of the table it appears that a considerable fraction of the disintegrations occur without the formation of the intermediate Be^8 nucleus, and even without the formation of the intermediate C^{12} and Be^9 nuclei in the (n, n') and (n, α) reactions respectively — i.e., some of the disintegrations are a direct breakup of the intermediate nuclei C^{12} or Be^9, or of the system C^{13}. This is confirmed by a comparison of the observed energy distribution of the reaction products (α particles and an inelastically scattered neutron)

Energy of the incident neutron	Fraction, in %, of the total number of disintegrations at a given energy which go through various levels of the intermediate nuclei.					
	Excitation energy of C^{12} (Mev)	%	Excitation energy of Be^9 (Mev)	%	Excitation energy of Be^8 (Mev)	%
9—11	<9.6	10			Ground state	40
11—14	level at 9.6	20	<11.6	12		
	>9.6	25			level at 2.9	10
14—17	level at 9.6	13				
	from 9.6 to 14	16	<11.6	4	Ground state	13
	>14.0	10			level at 2.9	8
17—19	level at 9.6	5				
	from 9.6 to 14	23	~11.6 level	4	Ground state	15
	from 14 to 15	5			level at 2.9	10
	>15	3				

in the center-of-mass system with a theoretical energy distribution calculated for the corresponding process by the statistical theory.[3] The comparison has shown that the fraction of direct break-ups increases with increasing energy of the primary neutrons.

A theoretical curve for the effective cross section has been calculated by Sachs,[4] who assumed that the system n + C^{12} broke up into a neutron and three α particles after statistical equilibrium had been reached within the interaction radius of n + C^{12}. This theoretical curve agrees well with the experimental one, which confirms the existence of direct break-ups. The agreement between the experimental and theoretical curves is particularly good in the first half of the energy range, the empirical coefficients characterizing the interaction volume being $R = 0.88$, $a = 1.06$ (Fig. 1).

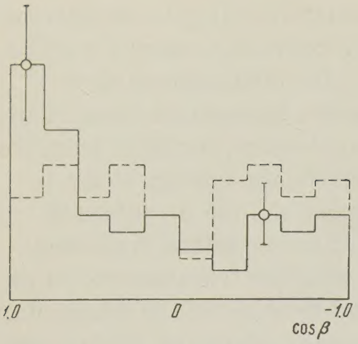

FIG. 2. The angular distribution of the inelastically scattered neutron in the center-of-mass system. Solid curve — $E_n = 18 - 19$ Mev, dashed curve — $E_n = 13 - 14$ Mev.

As the energy of the primary neutrons increases, the effective cross section for the process $C^{12} \rightarrow 3\alpha$ decreases. This can be partially explained by the presence of direct interaction processes, which occur without the formation of a compound nucleus. It is well known that a direct interaction is characterized by an anisotropic angular distribution for the inelastically scattered neutron (in the center-of-mass system), and the presence of reaction products with an anomalously high energy. For neutron energies E_n larger than 18 Mev, the center of mass angular distribution of the inelastically scattered neutron does have a peak in the forward direction (Fig. 2). In this same energy range, the energies of the inelastically scattered neutrons were anomalously high. This confirms the existence of a direct interaction for $E_n \geq 18$ Mev. These processes must, of course, decrease the effective cross section since the energy of the emitted neutron is, in many cases, too large for disintegration to take place.

For $E_n \geq 18$ Mev, there are also anisotropies in the angular distributions of the α particles. About 70% of the α particles from the disintegration fly out forward, which suggests that the incident neutron might just knock the α particle out of the C^{12} nucleus.

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EQUILIBRIUM DISTRIBUTION OF CHARGES IN A BEAM OF IONS OF LIGHT ELEMENTS

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The equilibrium distribution of ionic charges of light elements with atomic numbers Z between 5 and 10 was studied after passage of the ions through hydrogen, air, argon, and celluloid film with velocities ranging from 3.5 to $8 - 11 \times 10^8$ cm/sec. The mean ionic charge $\bar{i}$ was found to be from 10 to 30% larger after passage through the film than after passage through air. The values of $\bar{i}$ in gases were found to differ by 10 — 20%. The distribution of the ions in a beam with respect to $\bar{i}$ is approximately Gaussian.

1. INTRODUCTION

It is well known that when accelerated multiply charged ions pass through matter, the exchange of electrons between the ions and the atoms of the medium results in the establishment of an equilibrium distribution of charges which is dependent on the velocity of the particles. The study of the equilibrium distribution of charges is of considerable interest both for practical purposes and for the purpose of improving our understanding of the processes which occur when ions pass through matter in the region for which there is no rigorous theory. The results of such investigations can be used in the study of problems associated with the production of multiply charged ions at high energies, in determining the nature of the particles generated in nuclear reactions, etc.

Thus far only very incomplete experimental data have been available regarding the distribution of the ionic charges of light elements. The most systematic results have been obtained at energies ranging from 20 — 40 to 200 kev.[1,2] For ions with nuclear charge $Z \gtrsim 5$ this energy region corresponds to velocities $v < v_0 = e^2/\hbar$, at which the mean ionic charge $\bar{i}$ is many times smaller than Z. For higher energies data on the charge distribution exist for helium ions[1,3,4] passing through various substances, for lithium[5] and nitrogen[6,7] ions after passage through organic films and for individual energies of nitrogen ions in nitrogen,[8] of oxygen and neon ions in argon[9] and of nitrogen, oxygen, and fluorine ions after passage through organic films.[10] Theoretical calculations of the charge distributions and mean charges for ions with $Z > 2$ (Refs. 11 — 15) are approximate and are in need of experimental verification.

In the present work we have determined the equilibrium distribution of the ionic charges of light elements with atomic numbers Z from 5 to 10 after their passage through hydrogen, air, argon, and celluloid film with velocities ranging from 3.5 to $8 - 11 \times 10^8$ cm/sec, i.e., for $v \sim 1.5 - 5\, v_0$.

2. EXPERIMENTAL METHOD

The source of fast particles was a 72-centimeter cyclotron. In this work we used the following accelerated ions:

$$^{11}B^{+1,+2,+3};\ ^{13}C^{+2,+3};\ ^{14}N^{+2,+3};\ ^{16}O^{+2,+3} \text{ and } ^{20}Ne^{+2,+3}$$

A narrow ion beam extracted from the cyclotron and passing through slit 1 (Fig. 1) was deflected in the magnetic field H_1 and passed through the entrance channel 2, of cross section 1×0.1 cm^2 and length 3 cm, into the charge-exchange chamber of about 4.8 m length. Exit channel 3 had a length of 3 cm and cross section 0.5×0.1 cm^2. The particles passing through the chamber were analyzed by magnet H_2 and were then registered by counters 5, which were described in Ref. 7. The pressure in the charge-exchange chamber during measurements of the equilibrium charge distribution was approximately 70 times greater than the pressure in adjacent portions of the vacuum system and was varied from 5×10^{-3} to 8×10^{-2} mm Hg.

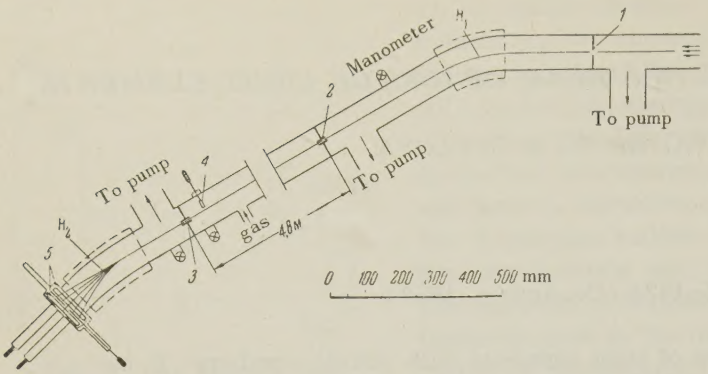

FIG. 1. Diagram of the apparatus

In the majority of cases the charge of the accelerated ions was smaller than the mean charge $\bar{i}$ of the equilibrium distribution. Therefore, when the pressure was increased, the mean charge at first increased and then began to decrease because the particles were decelerated by the gas. The maximum mean charge corresponded to the pressure at which the velocity loss of the particles passing through the charge-exchange chamber was about 2%. At this and higher pressures, the charges had practically an equilibrium distribution. The relative intensity Φ_i of any individual charge group in the ionic charge distribution (i) was determined by the velocity of the ions and varied little with the pressure. In most instances the error in values of Φ_i (Table I) was the possible result of ion scattering in the gas. The magnitude of this error was estimated from the variations of Φ_i with pressure. The statistical error was important only for $\Phi_i \leq 0.02$. Errors due to other causes were insignificant.

In order to determine the equilibrium charge distribution after the passage of ions through an organic film, film 4 of thickness ~ 10 μg/cm^2 was placed in the path of the beam, 2 cm in front of channel 3. After passage through this film the ions had an equilibrium distribution, as was shown by measurements with a film of twice the thickness. During the measurements with the film the pressure along the entire trajectory of the beam was ~ 1 — 2 $\times 10^{-5}$ mm Hg. The width of channel 3 remained 1 mm, as during the determination of charge distribution in gases. In some instances the distributions were compared as the width of the channel was varied from 0.3 to 3 mm. The variation of the relative intensities of individual charge groups then did not exceed in absolute value the variations which resulted from a 1% change of velocity.

TABLE I. Relative errors of experimental values of Φ

Φ	0.4	0.1	0.02	0.003
$\Delta\Phi/\Phi$	0.02	0.04	0.010	0.20

The velocity of the particles was calculated from the field strength of the deflecting magnet, which was calibrated with deuterons of known energy, making allowance for energy losses in the gases and film. The error in the velocity determinations was due mainly to inaccuracy in the calibration of the magnet and amounted to ~ 2%. The error in the velocity ratio did not exceed 1% and resulted from variation of the ion trajectories in the magnetic field of the cyclotron and from errors in determining the field of the deflecting magnet. In each individual experiment the velocity spread of the ions traversing channel 2 into the charge exchange chamber did not exceed 0.5%.

3. EXPERIMENTAL RESULTS

The measurements of the equilibrium distribution of boron, nitrogen, and neon ion charges are represented in Figs. 2 — 4. The values of Φ_i between the experimental points were taken from curves representing the dependence of $\log(\Phi_{i+1}/\Phi_i)$ on $\log v$, which were plotted from the experimental values of Φ_{i+1}/Φ_i; this is simpler than the dependence of Φ_i on v and permits more reliable interpolation. As an example Fig. 5 gives curves showing the dependence of $\log(\Phi_{i+1}/\Phi_i)$ on $\log v$ for oxygen ions. In order to obtain the dependence of Φ_i on v at low velocities for nitrogen and neon ions we used the experimental values of Φ_i in Ref. 2. For nitrogen, oxygen, and neon ions the error in the values of Φ_i between our experimental points does not exceed the errors in the measurements of Φ_i which were mentioned above. For boron ions, and also for nitrogen and neon ions at low velocities, the interpolation is less reliable and the error is apparently 2 — 3 times greater.

The velocity dependence of the degree of ionization $\bar{i}/Z$ in the different media is given in Fig. 6. The error in $\bar{i}/Z$ is about 1%.

In addition to the results already mentioned, we obtained the equilibrium distribution of helium ion charges at three different velocities, after passage through a celluloid film. These results are given in Table II.

Figures 2 — 4 show that the charge distribution in an ionic beam generally differs after passage through

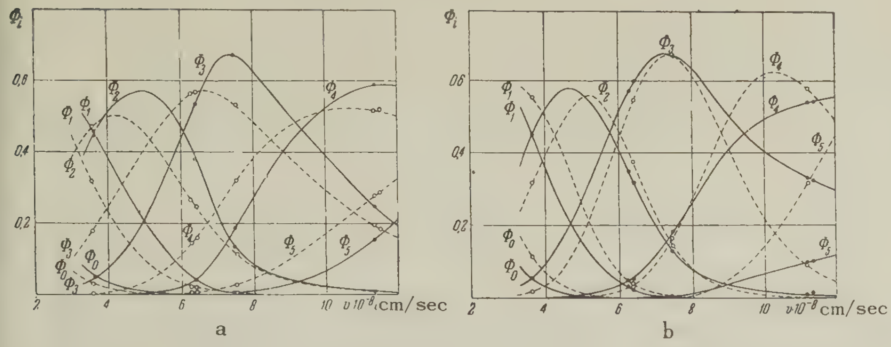

FIG. 2. Equilibrium distribution of boron ion charges after passage through a: ●— air, ○— celluloid film; b: ●— argon, ○— hydrogen

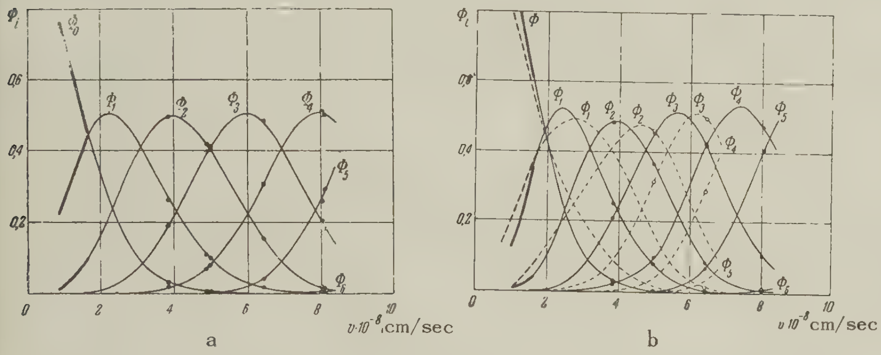

FIG. 3. Equilibrium distribution of nitrogen ion charges after passage through a: air; b: ●— argon, ○— hydrogen. The heavier lines at low velocities represent results taken from Ref. 2

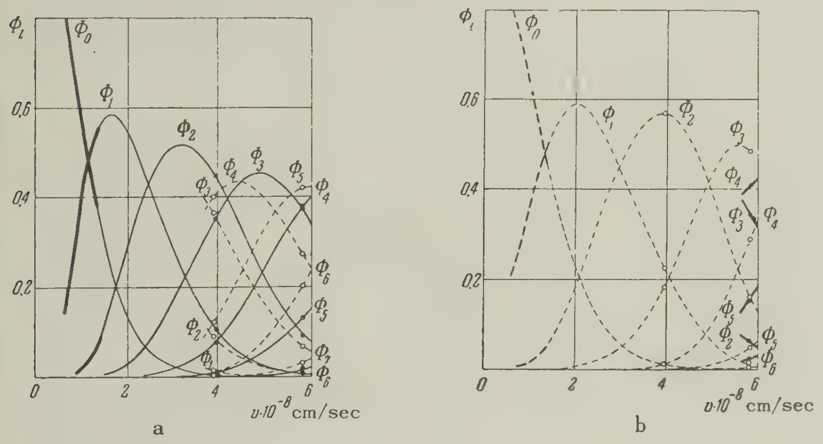

FIG. 4. Equilibrium distribution of neon ion charges after passage through a: ●— air, ○— celluloid film; b: ●— argon, ○— hydrogen. The heavier lines at low velocities represent results taken from Ref. 2

different media. There is a striking difference in the character of the velocity dependence of Φ_{i+1}/Φ_i after passage of the ions through solid and gaseous matter (Fig. 5). When boron and oxygen ions pass through a celluloid film, just as for ions of helium,[4] lithium,[5] and nitrogen,[7] the values of

$$k_i = \frac{d\log(\Phi_{i+1}/\Phi_i)}{d\log v}$$

vary little for velocities from 4 to 8×10^8 cm/sec, so that Φ_{i+1}/Φ_i shows a nearly power-law dependence on the velocity ($\Phi_{i+1}/\Phi_i \approx C_i v^{k_i}$). For all of the indicated ions $k_0 \sim 2.5 - 3$; for $i \sim Z/2$ the value of k_i is close to 4 and k_{Z-1} is around $4.5 - 5$. When boron, nitrogen, and oxygen ions pass through gases the values of k_i in the same velocity range are greatly increased. For example, in the case of oxygen in argon k_2, k_3, and k_4 increased from $3 - 4$ to $6 - 7$, which is almost double. At high velocities k_i is again reduced, as is shown by the data for boron ions.

$\log(\Phi_{i+1}/\Phi_i)$ has different values for different gases at the same ion velocity, but in most cases the differences $\log(\Phi_i/\Phi_{i-1}) - \log(\Phi_{i+1}/\Phi_i) = 0.434/\sigma_i^2$ are identical within the limits of error. σ_i does not change greatly as the charge and velocity are varied. The dependence of σ_i on v and i is found to be especially slight for $i < Z - 2$ in the case of nitrogen and oxygen ions. In the velocity range from 3.5 to 8×10^8 cm/sec the variation of σ_i for these ions is ± 7 — 10%. For $\sigma_i = \sigma = \text{const}$ we have the relation

$$\Phi_i = A\exp[-(i - i_0)^2/2\sigma^2],$$

where

$$i_0 = i + \frac{\sigma^2}{2}\ln(\Phi_{i+1}/\Phi_{i-1}).$$

For not very small values of the mean charge $\overline{i}$ the value of A is close to $1/\sigma\sqrt{2\pi}$, while i_0 is close to $\overline{i}$ [when $\overline{i} > 1.5$ and $\sigma \approx 0.7$, we have $|\overline{i} - i_0| < 0.003$ and $|(1/\sigma\sqrt{2\pi}) - A| < 0.002$]. Therefore

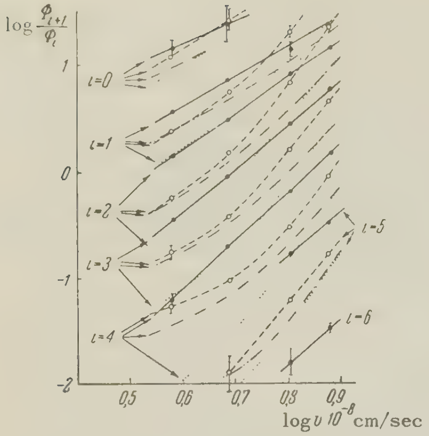

FIG. 5. Dependence of $\log(\Phi_{i+1}/\Phi_i)$ on $\log v$ for oxygen ions after passage through hydrogen (dots), air (dots and dashes), argon (dashes and open circles), celluloid film (solid lines and heavy dots).

our values of Φ_i can be represented approximately by the Gaussian function

$$\Phi_i(i-\bar{i}) \approx \Phi^0(i-\bar{i}) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(i-\bar{i})^2}{2\sigma^2}\right].$$

For boron, nitrogen, oxygen, and neon ions σ is 0.65, 0.80, 0.83, and 0.76, respectively. For nitrogen and oxygen ions with velocities from 3.5 to 7.5 × 10^8 cm/sec, the experimental values of $\Phi_i(i-\bar{i})$ do not differ from $\Phi^0(i-\bar{i})$ by more than 5% when $\Phi_i \geq 0.2$ (Fig. 7); when $\Phi_i \approx 0.1$ the difference is not greater than 15% and for $\Phi_i \sim 0.02$ it is about 30%. For boron ions in the velocity range 3.5 – 11 × 10^8 cm/sec the corresponding difference is 1.5 – 2 times greater and for neon ions it is 3 times greater than for oxygen ions. The values of Φ_i, after passage through a celluloid film, can also be represented approximately by a Gaussian function with the same values of σ as for the passage of the ions through gases (for carbon ions $\sigma \approx 0.73$). But the difference between $\Phi_i(i-\bar{i})$ and $\Phi^0(i-\bar{i})$ in this case is 1.5 – 2 times greater than for the passage of the same ions through gases.

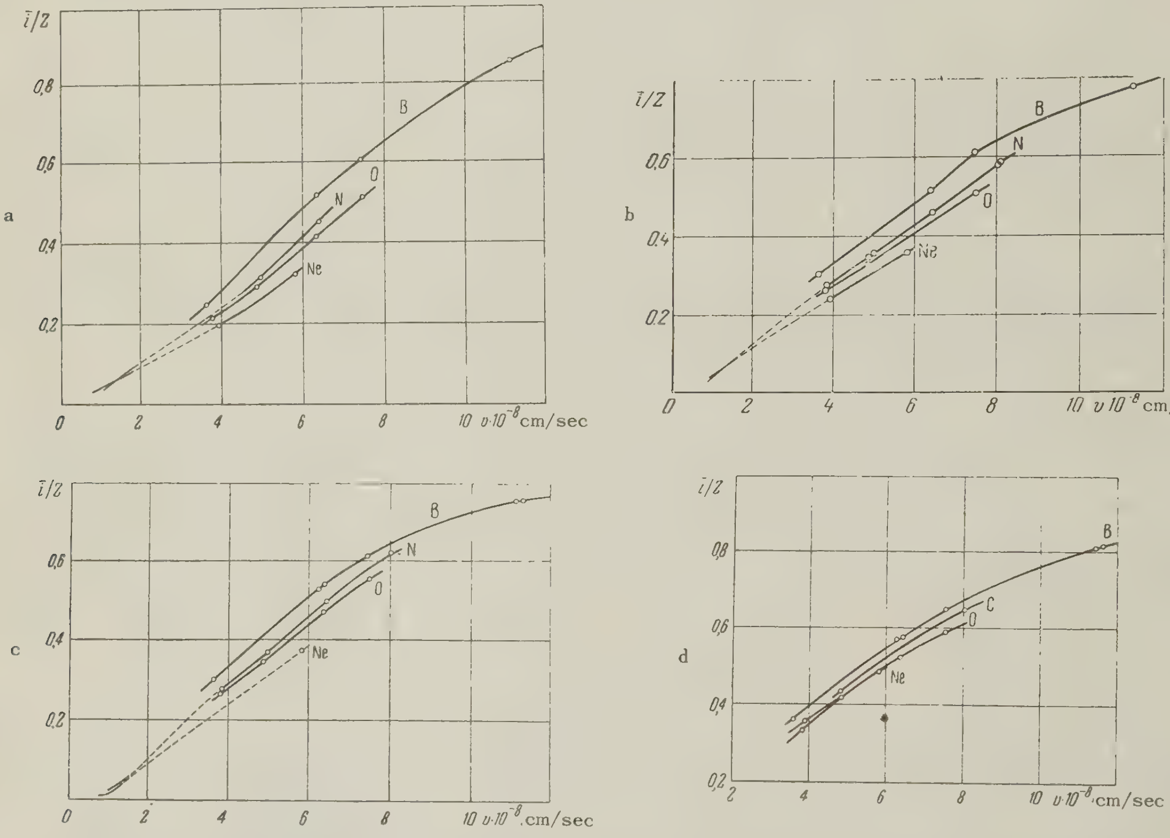

FIG. 6. The velocity dependence of the degree of ionization $\bar{i}/Z$ after the passage of ions through a — hydrogen, b — air, c — argon, d — celluloid film. The solid lines at low velocities represent results taken from Ref. 2

The following characteristics of $\bar{i}/Z$ can be noted for the passage of ions through different media:

1. For values of $\bar{i}/Z$ from 0.2 to 0.6 the mean charge in argon, for all of the ions investigated, is greater than the mean charge in hydrogen (by about 10 – 20%). The mean charge in air depends less strongly on velocity than the mean charge in hydrogen and argon. When $\bar{i}/Z \sim 0.3$ it is close to the

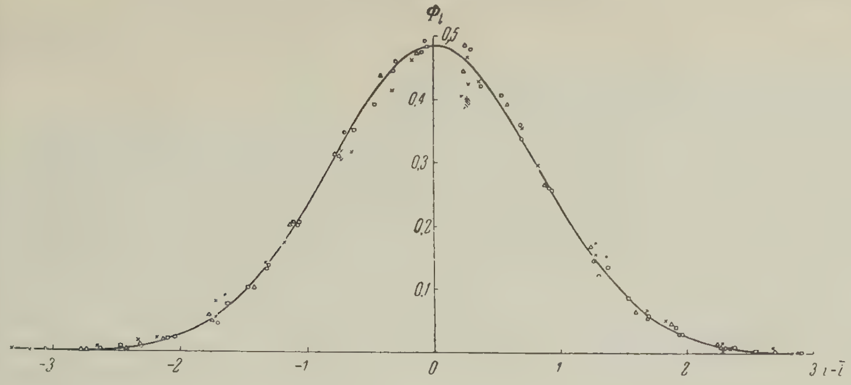

FIG. 7. Dependence of Φ_i on $i-\bar{i}$ for oxygen ions: ○ — in hydrogen, □ — in air, Δ — in argon, and × — in celluloid film. The solid line is a Gaussian curve with $\sigma = 0.83$

TABLE II. Φ_i and $\bar{i}$ for helium ions passing through celluloid film ~ 20 μg/cm² thick

$v \cdot 10^{-8}$ cm/sec	Φ_0	Φ_1	Φ_2	$\bar{i}$
6.0	0.017	0,368	0.615	1.60
8.0	0.002	0,145	0,853	1,85
18.3	~0,0001	0,005	0.995	1,995

mean charge in argon, but when $\bar{i}/Z \sim 0.5$ it approaches the value of $\bar{i}$ in hydrogen. For $\bar{i}/Z \sim 0.8$, the mean charge of boron ions is greatest in hydrogen, and smallest in argon. In hydrogen $\bar{i}$ is 10% greater than in air and 13% greater than in argon.

2. The mean charge of ions after passage through celluloid film in a broad range of velocities is greater than the mean charge of the ions in gases, and with increasing nuclear charge Z of the ions the excess of the mean ionic charge after passage through the film over the mean charge in air increases quite rapidly. While for boron ions with $\bar{i}/Z \sim 0.4$ this excess is ~ 15%, for neon ions it is ~ 30%. For the light group of uranium fission fragments ($Z = 38$) this difference is known[16] to be 40% and for the heavy fragments ($Z = 54$) it is 50%. Only at relatively high velocities with $\bar{i}/Z \sim 0.8$, for which data on boron ions are available, is the mean charge after passage through the film found to be 4% smaller than the mean charge in hydrogen.

3. The degree of ionization of the investigated ions in air, hydrogen, and argon can be represented within the range $\bar{i}/Z \sim 0.2-0.6$ as a different function of the parameter v/v_0Z^{α} with $\alpha \approx 0.4$ for each gas. When $\bar{i}/Z$ is represented in the same form for the passage of ions through a celluloid film, the exponent α is not even approximately constant: for $Z \sim 5-6$ the value of α is close to 0.35, while for $Z \sim 8-10$ we have $\alpha \sim 0$.

4. DISCUSSION OF RESULTS

Our values for the mean charge of ions which have traversed a celluloid film are in good agreement with the mean charge of nitrogen ions given in Refs. 6 and 7.

The distributions of nitrogen and oxygen ion charges which are given by Stephens and Walker[10] can be compared with the corresponding distributions in the present work and in Refs. 6 and 7 if the ion velocities given in Ref. 10 are reduced by 8%. In view of this discrepancy, Reynolds et al. checked their ion velocity measurements by measuring the energy of recoil protons. In the present work and in Ref. 7, ion velocities were determined from the curvatures of particle trajectories in the magnetic field as calibrated with deuterons, and they cannot contain such a large error. In Ref. 10 the velocities were calculated from the frequency of the accelerating electric field between the dees and from the final trajectory radius, and may therefore be too high.

The mean charge of oxygen and neon ions in argon, as given in Ref. 9, is 5 — 7% below the mean charge of these ions in argon as shown in Fig. 6c, and is about 7% greater than the mean charge in hydrogen. In Ref. 9 no essential difference was found in the values of $\bar{i}$ for different gases, but the authors do not exclude the possibility of a difference on the order of 10%. Within this limit of accuracy, the results of Ref. 9 agree with ours.

The article by Korsunskii et al.[8] gives the charge distribution only for the charge portion of a beam of nitrogen ions passing through nitrogen. Since the values of Φ_0 are not given, it is best to compare values of Φ_{i+1}/Φ_i. The values of Φ_2/Φ_1 for ion energies of 0.8 — 1.0 Mev, are approximately 1.5 times lower than in our present work. This discrepancy may possibly result from the fact that in Ref. 8 the beam passed through a layer of gas only about one tenth as thick as ours. Under such conditions the charge distribution can differ appreciably from equilibrium, as shown by our measurements. The values of Φ_3/Φ_2

and Φ_4/Φ_3 agree with ours; the explanation for this may be that for charges which differ more from the mean charge the equilibrium value of Φ_{i+1}/Φ_i is reached in a thinner layer of gas. In both investigations the values of Φ_5/Φ_4 are apparently not very accurate, but agree in order of magnitude.

Existing theoretical calculations of the equilibrium distribution and mean charge yield values of Φ_i and $\bar{i}$ which differ considerably from the experimental results. Calculations of the equilibrium charge distribution in Refs. 11 and 12 give lower values of k_i (around 3). The calculated values of Φ_i are at the maximum ~ 20% below experimental values. The mean charge as calculated in Refs. 11 – 14 differs by 10 – 30% from experiment, and none of the methods used agrees better than the others with experiment. Bohr's formula $\bar{i} = Z^{1/3}v/v_0$ in Ref. 17 gives values of $\bar{i}$ which are 1.5 – 2 times higher than experiment.

Brunings, Knipp, and Teller,[15] in attempting to improve Bohr's approximate formula, assumed that an ion traversing matter retains electrons with orbital velocities that satisfy $v_e > \gamma v$. For the calculation of v_e, they used the statistical model of the atom; in one calculation they separated from the electronic cloud of the ion the energetically most easily removable electron, and in another calculation they separated the outermost electron. The two different calculations are represented by the coefficients γ_1 and γ_2, respectively. For $\gamma_1 = \text{const}$ the dependence of $\bar{i}/Z$ on v in the interval $\bar{i}/Z \sim 0.3 - 0.6$ is stronger, and for $\gamma_2 = \text{const}$ it is weaker than the experimental dependence for all of the substances investigated. With increasing velocity, the experimental coefficient γ_1 is reduced by 10 – 30% while γ_2 increases by 10 – 30%. According to Ref. 15, the degree of ionization $\bar{i}/Z$ in the same range must be a function of v/v_0Z^{α}, where for $Z \sim 6 - 10$ with $\gamma_1 = \text{const}$ $\alpha \approx 0.55$ and with $\gamma_2 = \text{const}$ $\alpha = 0.33$. Experiments show that for ions passing through gases $\bar{i}/Z$ actually can be represented as a function of v/v_0Z^{α}, and although the form of the function differs with the substance we have the approximately constant value $\alpha \approx 0.4$ within the indicated limits. Bohr's formula gives $\alpha = 2/3$. Thus the consistent use by Brunings et al. of the statistical atomic model gives a more nearly correct relation between $\bar{i}/Z$ and Z than does Bohr's formula.

The method of Brunings, Knipp, and Teller does not permit us to draw any conclusions regarding the different values of $\bar{i}$ in various media. Livesey's calculation[13] of the mean charge of nitrogen ions in air and solid matter does not give the correct ratio between the values of $\bar{i}$ in these media. According to these calculations, the mean charge in a solid is smaller than the mean charge in air. A qualitatively correct ratio between the values of the mean charge in different gases with $\bar{i}/Z \sim 0.5$ was obtained by Gluckstern,[11] who calculated $\bar{i}$ from the cross sections for electron loss and capture. However, the absolute values which he obtained for $\bar{i}$ are 20 – 30% below the experimental values.

Our relations between the values of the mean ionic charge in different media reflect the behavior of the cross sections for electron capture σ_c and loss σ_ℓ in these substances. The cross sections σ_c and σ_ℓ increase with the atomic number Z_m of the medium. Larger values of $\bar{i}$ in heavier gases for $\bar{i}/Z \sim 0.3 - 0.5$ ($v \sim 4 - 7 \times 10^8$ cm/sec) show that σ_ℓ increases with Z_m by a higher power than σ_c, and this is a qualitative confirmation of the approximate determination of σ_ℓ and σ_c based on the statistical model. According to these calculations[17,18] σ_ℓ is proportional to $Z_m^{2/3}$ and σ_c is proportional to $Z_m^{1/3}$. At high velocities σ_ℓ must be proportional to Z_m^2 and σ_c must be proportional to Z_m^5, so that in the lightest elements the mean charge must be larger. We obtained the corresponding ratio between the values of $\bar{i}$ in gases for boron ions with $v \sim 11 \times 10^8$ cm/sec. The relative increase of the mean ionic charge in air compared with hydrogen and argon for $v \leq 4 \times 10^8$ cm/sec cannot be explained by the Thomas-Fermi statistical model and is apparently associated with the structural characteristics of atoms with unfilled shells, especially in the case of relatively large atomic dimensions, which must result in some increase of σ_ℓ.

The large value of $\bar{i}$ in solid matter compared with its values in gases for uranium fission fragments has been attributed by Bohr and Lindhard[19] to changes in σ_ℓ and σ_c resulting from ionic excitation. An analogous difference between values of $\bar{i}$ for lighter elements evidently has the same causes. A reduction of the mean electronic binding energy in an ion traversing solid substances must result not only in an increase of σ_ℓ but also evidently in a shift of the maximum of σ_ℓ as a function of v at low velocities. Since the cross sections for the simultaneous loss and capture of two electrons are considerably smaller than the corresponding cross sections for a single electron[18,20], for the most intense charge groups, at least, $\Phi_{i+1}/\Phi_i \approx \sigma_\ell(i)/\sigma_c(i+1)$, where $\sigma_\ell(i)$ and $\sigma_c(i+1)$ are the single-electron loss and capture cross sections for ions with charges i and i + 1, respectively. Thus

$$k_i = \frac{d \log (\Phi_{i+1}/\Phi_i)}{d \log v} \approx k_l(i) - k_c(i+1),$$

where

$$k_l(i) = d\log\sigma_l(i)/d\log v, \quad k_c(i+1) = d\log\sigma_c(i+1)/d\log v.$$

The shift of the maximum of σ_l towards lower velocities when ions pass through solids must lead to a reduction of k_l and consequently of k_i; this is observed experimentally. Thus the difference in the equilibrium charge distribution after ion passage through gases and film can be explained qualitatively by excitation of the ions. A more thorough discussion of this question requires more complete information concerning the cross sections for electron loss and capture.

In conclusion we consider it our duty to thank S. S. Vasil'ev for his comments during the discussion of our results, and to the cyclotron crew, especially G. V. Kosheliaev, A. A. Danilov, and V. P. Khlapov, for their assistance in the experimental work.

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ANGULAR CORRELATION BETWEEN FRAGMENTS AND CHARGED PARTICLES EMITTED IN URANIUM FISSION

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Fission events induced by 660-Mev protons in uranium nuclei, involving the emission of charged particles, have been observed in P-9 nuclear emulsions. The angular distribution of these particles, with respect to the direction of the incident beam and the fragment line of flight, has been measured. It has been established that the protons do not exhibit an angular correlation with the fragments; however, the possibility has not been excluded that some of the protons are emitted by the fission fragments. The α particles and multiply-charged particles have a tendency to be emitted at large angles with respect to the fragments. Possible factors which may be responsible for the observed correlation between the fragments and α particles are discussed.

INTRODUCTION

EXPERIMENTS on fission in heavy nuclei induced by high-energy nucleons, performed with nuclear emulsions, indicate that fission is frequently accompanied by the emission of protons, α particles, and heavier particles, which are emitted from the fission site. In a photoemulsion, such a disintegration event looks like a star containing two additional dense fragment tracks ("star" fission).[1-4] The existence of an anisotropy with respect to the beam direction (approximately 60 percent of the particles are emitted in the forward hemisphere) leads us to assume that some of these particles are emitted by the nucleus in the first or cascade stage of the disintegration process. An estimate of the fraction of cascade protons, based on the magnitude of this anisotropy, is in agreement with the results of a Monte Carlo calculation. The isotropic distribution of low-energy charged particles can, in most cases, be associated either with evaporation from the nucleus prior to fission or emission from excited fragments. Comparing the number of charged particles observed in fission of heavy nuclei with the theoretical number expected on the basis of a model in which there is preliminary emission by the fissioning nucleus, Shamov[3] concludes that the α particles and protons are evaporated from the nucleus prior to fission. In a paper by Ivanova and P'ianov,[5] however, it is indicated that a comparison of this kind does not allow us to distinguish emissive uranium fission from a fission which occurs in one of the intermediate levels of an excited nucleus. A calculation carried out by these authors has shown that the sum of the number of charged particles emitted by the uranium nucleus in cooling to approximately half the initial excitation energy, plus those emitted by the fragments, is equal to the number of particles which would be emitted from the urnaium nucleus before it is completely cooled prior to fission.

One of the factors that permit appraisal of the validity of various fission models at high excitation energies is the existence or absence of an angular correlation between the paths of the evaporated particles and the fission fragments.[6] An experiment on correlation of neutrons and fragments, similar to the well-known Fraser experiment on thermal neutrons, was carried out by Harding and Farley, who investigated uranium fission induced by 147-Mev protons.[7] Harding and Farley concluded that the majority of neutrons (approximately 80 percent) are emitted by the nuclei prior to fission. However, this result does not exclude the possibility that an increase in the energy of the bombarding particles to 500 — 700 Mev might result in the development of an intranuclear cascade and that the increased excitation energy might lead to a situation in which some of the highly-excited nuclei undergo fission from one of the higher levels.

At the present time, there is apparently only one paper on the angular distribution of light charged particles with respect to the fragments in fission induced by fast nucleons.[4] In a paper by Belovitskii et al.[4] a study was made of uranium fission induced by neutrons obtained by charge-exchange from 480-Mev protons. It was found that protons were emitted isotropically, whereas the α particles have a tendency to be emitted at right angles to the fragment line of flight. Unfortunately, because of poor statistics (27 α particles) this result was not very conclusive.

In the present work we have undertaken to investigate, with better statistics, the angular distribution of charged particles with respect to fragments in uranium fission by high-energy protons. Since an angular correlation is to be expected only for particles of evaporation origin, the investigation was carried out with a low-sensitivity emulsion which could be exposed to a higher flux of bombarding protons.

EXPERIMENTAL ARRANGEMENT

In this work use was made of plates with P-9 fine-grain nuclear emulsion prepared in the laboratory of Professor N. A. Perfilov. The thickness of the sensitive layers was 100 and 200μ. The emulsions were sensitive to 30-Mev protons. The uranium was introduced by impregnating the emulsion in a water solution of $NaUO_2(C_2H_3O_2)_3$.

The plates were exposed in the extracted 660-Mev proton beam from the synchrocyclotron of the Joint Institute for Nuclear Research. Some of the plates (first series) were exposed in a proton flux which was parallel to the plane of the emulsion; the second group was exposed perpendicularly to the beam. For protection against slow neutrons the plates were placed in a cadmium cover with double walls, between which borax was placed (a layer 1 cm thick). The emulsions were processed in the usual way, using temperature development. The scanning was carried out with a MBI-3 microscope with a magnification of 5 × 90 × 1.5; the measurements were performed at maximum magnification. The angles were measured with an ocular angle gauge accurate to 1° — 2°; the accuracy of the length measurements in the vertical and horizontal projections was approximately 0.5μ.

The first series of plates yielded the spatial angular distribution of the particles with respect to the fragments. For this purpose measurements were made of the inclination of each track to the plane of the emulsion and the angle between the projections of the tracks on this plane. The shrinkage factor for the emulsion was determined as by Vigneron[8] and Gramenitskii and Podgoretskii[9] from the observation of the α particles due to the natural radioactivity of the uranium in the same plates. The shrinkage factor was found to be 1.85 ± 0.15. The same value was obtained by direct measurement of the emulsion thickness before and after development, using other plates under analogous experimental conditions.

In the second series of plates, which were exposed perpendicularly to the beam, only the angular distribution in projection was taken. In fission induced by high-energy particles the fragments frequently fly apart at angles less than 180° so that in these cases the angles could be read by using the lines connecting the ends of the fragment tracks.

The identification of the tracks of charged particles in "star" fission events was carried out visually. The reliability of this method of distinguishing singly and doubly charged particles (with preliminary training of the observers) has been indicated by Serebrennikov[10] and was verified in the present work. The error in assigning tracks to one or the other of the groups was less than 5 percent. Doubtful cases were not included in the experimental statistics.

EXPERIMENTAL RESULTS

In all, 3201 cases of uranium fission were recorded, including 1156 in the plates of the second series. The prong distribution of the fissions (i.e., the distribution over the number of light charged particles emitted in fission) was as follows: 0 prongs — 43.7%, 1 prong — 25.3%, 2 prongs — 15.7%, 3 prongs — 8.5%, 4 prongs — 4.2%, 5 prongs — 1.5%, 6 prongs — 0.8%, 7 and more prongs — 0.3%. The ratio of the number of α particles to the number of protons was 0.35 ± 0.02. The angular distributions of these particles with respect to the direction of the proton beam are shown in Figs. 1 and 2. The proton distribution is in agreement with the results obtained by Shamov[3] as far as the front—back ratio is concerned; however, α-particle distribution obtained in Ref. 3 is isotropic, the present data indicate that a considerable fraction of the α particles are to be associated with an anisotropic emission process.

The table lists the particle distribution over the angle formed by the particle path and fragment path (projected on the plane of the emulsion), under various exposure conditions. The last row of this table contains the anisotropy coefficient, defined as the ratio of the number of tracks lying in the interval 60 — 90° to the number of tracks at angles 0° — 30° with respect to the fission line. Statistical errors are indicated. In the last column are given data on particles with charge greater than 2. Some of the tracks assigned to the triply charged particles may actually be those of α particles, mistaken because of the inadequate visual resolution between these particles. In Figs. 3 and 4 are given the spatial angular distributions of the particles, as obtained by measuring tracks in plates of the first series and converting to unit solid angle.

Projected fragment angle (degrees)	Parrallel exposure		Perpendicular exposure		Total number of particles		
	Protons	α particles	Protons	α particles	Protons	α particles	Protons with $Z \geqslant 3$
0—15	233	85	230	67	463	152	4
15—30	242	81	210	61	452	142	2
30—45	274	87	234	77	508	164	4
45—60	260	87	213	78	473	165	2
60—75	264	108	210	86	474	194	7
75—90	245	97	214	97	459	194	8
$\frac{N_{60-90°}}{N_{0-30°}}$	1.06 $\pm$ 0.09	1.23 $\pm$ 0.18	0.96 $\pm$ 0.09	1.43 $\pm$ 0.24	1,02 $\pm$ 0,07	1.32 $\pm$ 0.14	2,5 $\pm$ 1.6

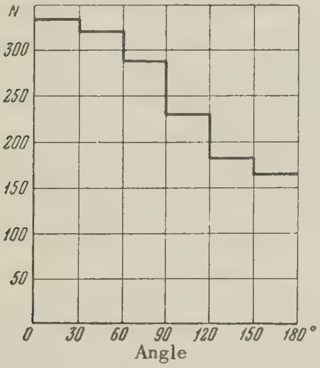

FIG. 1

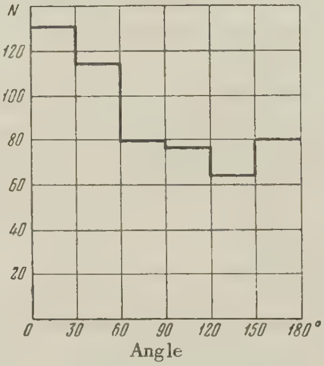

FIG. 2

FIG. 1. Angular distribution of protons with respect to the beam in star fissions (projection).

FIG. 2. Angular distribution of α particles with respect to the beam in star fissions (projection).

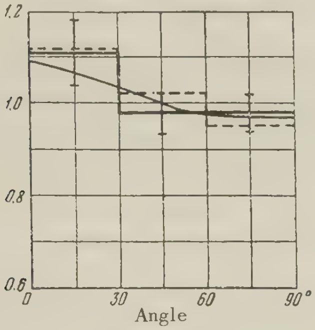

FIG. 3

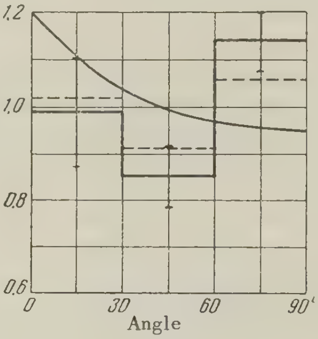

FIG. 4

FIG. 3. Angular distribution of protons with respect to fragments, reduced to unit solid angle: the dashed line is the measured distribution; the solid line is the corrected distribution; the smooth curve is calculated from the barrier-fission model.

FIG. 4. Angular distribution of α particles with respect to fragments, reduced to unit solid angle: the dashed line is the measured distribution; the solid line is the corrected distribution; the smooth curve is calculated from the barrier-fission model.

DISCUSSION OF THE RESULTS

A comparison of the data of the table and the distributions shown in Figs. 3 and 4 by the dotted lines indicates a certain discrepancy. Figure 3 seems to indicate a preference for emission of protons in the direction of the fragments, whereas the measurements of angles in projection on the same plates do not corroborate this finding. The origin of this discrepancy may be in errors arising in taking the spatial distribution or errors in the shrinkage factor. In order to establish the effect of the shrinkage factor on the shape of the spatial distribution the latter was plotted a second time, using 400 arbitrarily chosen tracks of protons from star fissions and taking a new value of the shrinkage factor, 2.2. No noticeable systematic change in this distribution was found. There is still one other factor which has a bearing on the shape of the angular distribution of the tracks. The spatial distributions of the fragments and the α particles due to natural radioactivity observed under the same conditions indicate that some of the tracks (approximately 25 – 30 percent) are at large angles (60° – 90°) with respect to the plane of the field of view and are lost in scanning the plates. Since the light charged particles have a certain directivity in the forward direction, even in the absence of any physical correlation between these particles and the fragments the experimentally determined distribution will be anisotropic with respect to the fragment line of flight. It is easily demonstrated that there will be an excess of cases exhibiting small angles between the track of the charged particles and the fragment. An estimate based on the data of Figs. 1 and 2 and on the observed fragment angular distribution with respect to the beam shows that the coefficient for this expected parallel anisotropy is about 1.05.

A comparison of the angular dependence for particles emitted in the given hemisphere, in which the directivity is less pronounced, with the corresponding dependence in the backward hemisphere corroborates the "independence" effect. In the observation of angles in the plane of the field of view this effect vanishes since the angular distribution of the acceptable cases is isotropic.

It may be assumed that the presence of cascade particles, directed predominantly along the beam, and the perpendicular anisotropy for uranium fission fragments with respect to this direction, also lead to a certain angular (perpendicular) correlation between the light particles and the fragments. Using the data of the present work (Figs. 1 and 2) and that of Ref. 11, in which the fragment angular distribution was obtained, we have computed the correlation associated with the superposition of the two distributions. The calculation was carried out numerically using the relation

$$\Phi(\varphi) = \int_{-\pi/2}^{\pi/2} f(\alpha)\, F(\varphi - \alpha)\, d\alpha,$$

where $\Phi(\varphi)$ is the angular distribution of the prongs with respect to the fragments, $f(\alpha)$ and $F(\varphi - \alpha)$ are the distribution functions for the fragments and charged particles respectively with respect to the beam direction; the first function is taken from Ref. 11 while the second is based on the data in Figs. 1 and 2.

It would appear that for protons the perpendicular anisotropy factor is not greater than 1.03; for α particles it is not greater than 1.02. It is obvious that this effect should not be observed in the perpendicular-exposure experiment. This coincidental correlation was actually not observed for protons in the plates of the second series; for α particles, however, the perpendicular-directivity effect was observed.

In Figs. 3. and 4 the solid lines show the spatial distributions with the above-mentioned factors taken into account. A comparison was made of the experimental distribution and that to be expected on the barrier-fission model. In Figs. 3 and 4 the solid curves show the calculated results (normalized to the number of experimental tracks). In carrying out the calculation it was assumed that fission occurs symmetrically starting at an initial excitation energy of 150 Mev which is divided equally between the two fragments. The mean fragment temperature at which charged-particle emission occurs is 2 Mev. The particle velocities were determined from the energy spectra, which were plotted taking account of the thermal expansion of the nucleus and barrier penetrability.[12] The kinetic energy of the fragments was taken as 80 Mev. The deviations from 180° in the direction of flight were not taken into account in the calculation.

Although the distributions shown in Fig. 3 and the results of the observations of the plates in the second series indicate the existence of a certain parallel anisotropy for the protons with respect to the fragments, it is felt that this effect cannot be attributed to the barrier mechanism of uranium fission since the departures from isotropy are within the limits of the statistical fluctuations. It is possible that a certain number of protons are emitted by the fragments, this is not inconsistent with the available data on the energy spectrum. As far as the α particles are concerned, the picture is somewhat more definite, since emission of these particles by fragments is not very probable and a perpendicular anisotropy is observed.

In experiments on uranium fission by slow neutrons it has been established that α particles with energies on the order of 20 Mev are emitted in a direction close to 90° with respect to the fragment line of flight with a probability of 1 in 400.[13 – 14] With an increase of the bombarding-neutron energy to 14 Mev the emission of α particles becomes still less probable[4] as an explanation for the correlation observed in the present work. It is possible that in fission induced by high-energy protons there is an effect analogous to the emission of α particles in fission at low excitation energies, with the sole difference that we are not dealing with the effect due to two individual fragments of the fissioned nucleus which are still fairly close, but rather two parts of a highly deformed elongated nucleus which is in a pre-fission stage. The effect of the electric field of these parts is of greater importance for particles with higher charge and hence the perpendicular directivity is more pronounced for α particles than for protons. This mechanism applies only for part of the particles emitted prior to fission because some of the particles are emitted at an instant when the nucleus is not deformed; these particles are not correlated with the oscillation directions of the nucleus.

In conclusion we may note that the observation of an angular correlation of this type is most conveniently carried out with perpendicular exposure since in this case the spurious factors associated with the directivity of the cascade particles are minimized.

The authors are indebted to N. A. Perfilov for valuable advice and his interest in this work, to N. S. Ivanov for proposing this research and participating in the discussion of the results, and to N. R. Novikov, E. I. Prokof'ev, and E. V. Fadin for preparation of the plates with which the experiment was performed.

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MEASUREMENT OF SURFACE TENSION AT THE BOUNDARY BETWEEN SUPERCONDUCTING AND NORMAL PHASES

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The surface tension between superconducting and normal phases has been determined by measuring the period of a simple laminar structure, obtained in a flat sample located in an inclined magnetic field. For a single crystal of tin, the results of measurements in the temperature interval 2.165° K to 3.5° K can be described by the relation $\Delta = 2.5 \times 10^{-5}(1 - T/T_c)^{-1/2}$ cm.

A theoretical investigation of the structure of the intermediate state was first carried out by Landau[1] as early as 1937. For a sample in a transverse magnetic field, the model of the structure which he proposed and regarded as the simplest was one in the form of alternating layers of superconducting and normal phases (s- and n-phases), a cross-section of which is shown in Fig. 1. The size of the layers in this model is related to the sample thickness and to the magnitude of the surface tension at the phase boundary by the expression

$$\alpha (8\pi/H_c^2) = (a^2/L)\,\varphi(\eta), \tag{1}$$

where H_c is the critical field, φ is a dimensionless function whose numerical value has been calculated elsewhere,[2] and $\eta = a_n/a$; the meaning of the remaining symbols is given in the figure. The quantity $\alpha \times 8\pi/H_c^2$, which has the dimensions of length, is usually designated as Δ.

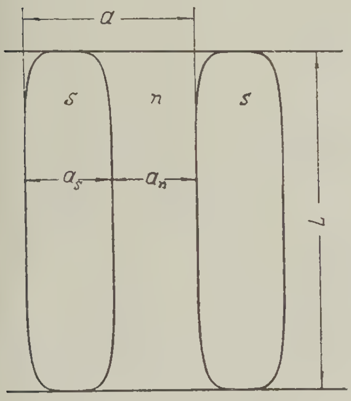

FIG. 1

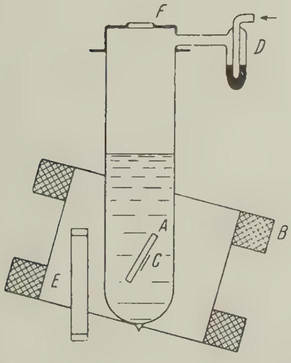

FIG. 2. Schematic diagram of the apparatus

A determination of the geometrical size of the s- and n-domains thus affords the principal opportunity for calculating the quantity Δ. An experimental investigation of the structure of the intermediate state by other methods has shown, however, that the shapes of the s- and n-domains are usually very complex, and far removed from the simple cases studied theoretically. Furthermore, the irregularity and irreproducibility of the structures has indicated that the intermediate state of the actual samples was not an equilibrium one, whereas the theoretical study presupposed thermodynamical equilibrium.

We have undertaken a search for experimental conditions under which the structure would have a simple form, permitting mathematical treatment, and for which it would be possible to obtain information concerning the degree of approximation of the structure to an equilibrium state.

Balashova,[3] while studying the structure of the intermediate state of a tin sphere by using a ferromagnetic powder, noticed that over a small region of the surface of the sphere, where the external field made an acute angle with it (i.e., near the magnetic equator of the sphere), the picture was much simpler than over the rest of the surface, and the n-domains had the form of comparatively wide, straight zones in the meridional direction, independent of the method by which the specimen was carried over to a given point of the region of the intermediate state on a diagram in the variables H and T. It was presently shown by Dzialoshinskii[4] that this alignment of the laminae is energetically the most favorable one. Proceeding from this, we have carried out a study of the structure of the intermediate state for a sample located at an acute angle β with respect to an external magnetic field.

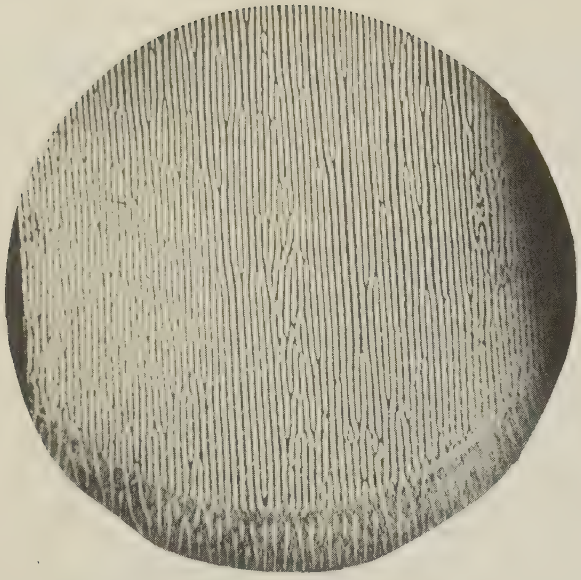

FIG. 3. View of the surface of the disc 50 mm in diameter (sample No. 1). T = 2.165° K, h = 0.950; $\eta = 0.62$, a = 0.54 mm.

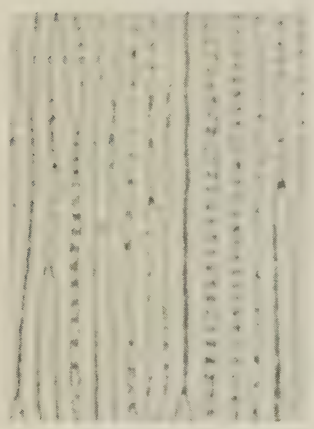

FIG. 4

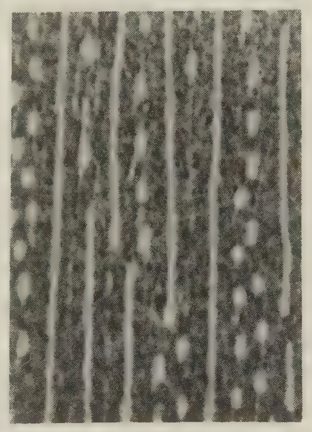

FIG. 5

FIG. 4. A portion of the surface of sample No. 1 (visual field is 7.3 mm wide). T = 2.165° K, h = 0.910, $\eta = 0.5$.

FIG. 5. A portion of the surface of sample No. 1 (visual field is 12 mm wide). T = 2.165° K, h = 0.995.

After preliminary experiments, we settled upon a sample in the form of a disc 50 mm in diameter and 2 mm thick, at an angle $\beta = 15°$. A schematic drawing of the experimental configuration is shown in Fig. 2. The sample A is located in the field of the coil B, fixed to an axis perpendicular to the plane of the drawing. Initially the field of the coil was directed along the surface of the sample, which was ascertained from a maximum in the conductivity of a thin tin film C, mounted parallel to the surface of the sample, after which the coil was turned through an angle β. The accuracy of the setting amounted to several minutes. The field of the earth was compensated by Helmholtz coils, not shown in the figure.

The structure of the intermediate state was exhibited with the aid of a fine nickel powder, blown into the Dewar flask from the atomizer D.[3] The surface of the sample was illuminated by the lamp E, and was viewed through the optical aperture F with the aid of a telescope with a reading device, or was photographed by a camera with a chamfered plate.

In Fig. 3 is shown one of the photographs taken of sample No. 1, which was a single crystal of pure tin (resistance at helium temperatures $\rho_{4.2°K} = 9 \times 10^{-5} \rho_{20°C}$). Over most of the surface of the disc the structure is satisfactorily close to a system of parallel layers. In spite of the presence of a certain amount of branching, the average period of the structure, determined over a length of 20 — 25 mm in various portions of the disc, turns out to be constant to an accuracy of 3 — 4%.

The region of irregular structure, with a large percentage of the n-phase in the lower part of the surface, is separated from the remaining part of the picture by a clear boundary, which represents the projection of the edge of the reverse side of the disc, brought about by the magnetic field penetrating the sample at an angle.

A structure of simple form was obtained only in a definite range of variation of the relative magnetic field h ($h = H_0/H_c$, where H_0 is the external field). For h < 0.91 — 0.92 (for T = 2.165°) and for h < 0.8 — 0.85 (for T = 3.2 — 3.5°), a complex structure arose. In Fig. 4, for example, it is quite apparent that small inclusions of the s-space have developed inside the n-domains.

For fields approaching the critical field (h > 0.99), the laminar structure is gradually replaced by a system of s-filaments. In Fig. 5 is shown an intermediate stage, in which filaments and laminae occur together.

For a laminar structure of the type shown in Fig. 3, which has a definite period, the quantity Δ can be determined from the experimental data, since the results calculated by Landau[1] can be used also when there is an inclined magnetic field. It can easily be shown that the geometrical shape of the layers remains the same as for a perpendicular field (Fig. 1), but Eq. (1) becomes

$$\Delta = (a^2/L)\,\varphi(\eta)\,(1 - h^2\cos^2\beta), \tag{2}$$

where

$$\eta = a_n/a = h\sin\beta/\sqrt{1 - h^2\cos^2\beta}.$$

Thus, to determine Δ it is necessary to know not only a, L, and ρ, but also the quantity h, with possibly greater accuracy since the difference $1 - h^2\cos^2\beta$ is small in our case. Because of this, H_c was determined in each experiment upon the disappearance of the last traces of superconducting filaments (with an accuracy of approximately 0.2%).

The principal measurements were carried out with sample No. 2, a single tin crystal of greater purity than sample No. 1 ($\rho_{4.2°K} = 1.4 \times 10^{-5}\,\rho_{20°C}$). The crystalline axis of 4-fold order lay in the plane of the specimen and was directed along the projection of the magnetic field onto the plane of the sample. In Fig. 6 we plot the results of one series of measurements at T = 3.4°. With the aim of showing up the influence of non-equilibrium of the state on the results, different methods were used to carry the specimen over to the point of the intermediate state under study: the field was decreased at constant temperature (whereupon it was necessary for new layers to arise and for the period a to diminish), or the field was increased. In all cases laminar structures developed, aligned in the direction of the projection of the field. It was also found, however, that as a rule the values of a exhibited an hysteresis, which was more noticeable for the higher values of η. The actual value of Δ ought to lie in between the results obtained for these transition modes. Transitions achieved by varying the temperature at constant field gave results analogous to those for a changing field. We also tried to decrease the influence of hysteresis by causing a transition by varying the field about the required value (points marked with small circles); the field was first changed by about 1%, and then the amplitude of oscillation was gradually decreased to zero. The period of oscillation amounted to 30 — 60 seconds, which was sufficient for the structure to be able to follow the changes of the field.

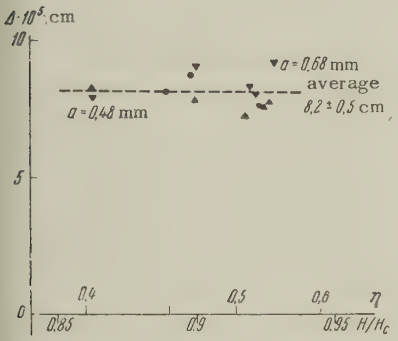

FIG. 6. Sample No. 2, T = 3.4° K. ▼ — decreasing field, ▲ — increasing field, ● — diminishing oscillating field.

It is difficult to make any definite statement concerning the origin of the hysteresis effects in our experiments. In every case chemical impurities, and in general those dislocations of the periodic lattice for which a strong effect shows up in the magnitude of the residual electrical resistance, apparently did not play an important role, since there was no improvement observed in going from sample No. 1 over to sample No. 2, which had a considerably smaller residual resistance. The surfaces of the samples were mirror smooth and probably free from strain, since the cast specimens separated quite easily from the mold. (The mold was made of glass, whose polished surface, next to the casting, was covered first with a film of oil a few molecular layers thick and then with a barely noticeable layer of carbon black.)

The absence of the requisite number of nucleii for one phase or the other cannot be invoked to explain the hysteresis, since on every picture there were observed several branches, i.e., nucleii virtually already formed.

Except for the hysteresis effects, the principal source of error in our measurements appears to be the fact that the specimen was not the infinite lamina to which Eq. (2) is strictly applicable. We can estimate the magnitude of this error by also carrying out a calculation in which we take the specimen to be an ellipsoid with an axis ratio of $1/25$. The data obtained for Δ are in this case smaller by 5 — 10% (in relation to the quantity h) when compared with the reduced values. However, since it is difficult to establish which method of calculation gives the most accurate results for the disc, we have not introduced a correction for the finite size of the sample, and have made use of Eq. (2). A detailed study of this question would make sense only if it were possible to make a considerable improvement in the reproducibility of the results.

In Fig. 7 are shown the results of measurements made at different temperatures. The uncertainty in the values of Δ can be estimated as about 6 — 8% on the side of increasing Δ, and 12 — 15% on the side of decreasing Δ. The results are most accurate at low temperatures. The asymmetry in the magnitude of the error is related to the fact that we have neglected to make the correction discussed above for the finite size of the disc. As is clear from the figure, within the limits of this error our results are described by the simple relation

$$\Delta = \frac{2.5 \cdot 10^{-5}}{\sqrt{1 - T/T_c}}. \tag{3}$$

The results of measurements on sample No. 2 agree well with the preliminary data obtained with sample No. 1. It should be emphasized that Eq. (3) is an asymptotic expression, valid only near T_c, and cannot be extrapolated to the region of much lower temperatures.

We also carried out a series of measurements with specimen No. 2 at T = 3.4° for different orientations of the axis of 4-fold order relative to the magnetic field (the sample could be turned about in its plane during an experiment). Within the limits of error, the results did not depend on the angle of deflection. It should be pointed out that these measurements were in the nature of a control experiment, and do not provide essential information concerning the anisotropy of Δ, since changes of the angle between the superconductivity current (directed near the axis of rotation) and the crystalline axes are negligible here.

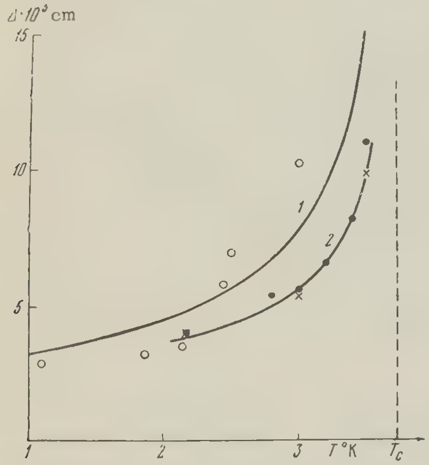

FIG. 7. ● — sample No. 2; × — sample No. 1. ○ — Shawlow's data. Solid curves: 1 — from theory of Ginzburg and Landau; 2 — from Eq. (3).

From the experimental data concerning Δ obtained recently by other authors, we have plotted in Fig. 7 the values obtained by Shawlow,[5] who studied the structure of the intermediate state with niobium powder. Shawlow's method has, however, a fundamental objection to it, and in our opinion his data are not sufficiently reliable. Shawlow ignored the appearance of a fine structure in the layers which he observed, apparently because the powder he used to bring out the pictures was too coarse. The specimens which he investigated were polycrystalline ones, with which appreciable hysteresis effects are associated, and one cannot estimate the magnitude of this hysteresis because placing the superconducting laminae perpendicular to the field, as Shawlow did, makes it impossible to obtain a simple well-ordered picture for the different transition modes. It turns out also to be impossible to use a single crystal in this case, as mentioned by the author himself.

A preliminary study of the structure of the intermediate state under various conditions[3] forced us eventually to reject Shawlow's method, which was, at first glance, the most natural one.

Certain information concerning the order of magnitude of Δ and its temperature dependence has been obtained by indirect methods. Shal'nikov,[6] the first to arrive at an estimate of the magnitude of Δ, used a bismuth probe to study the magnetic field in gaps of different widths formed between tin hemispheres. His estimate for Δ amounted to $0.6 - 1.5 \times 10^{-5}$ cm for T = 3.1° K. However the theoretical assumptions concerning the degree of uniformity of the magnetic field over the width of the gap, on which his work was based, can nowadays be regarded as justified only qualitatively.

Desirant, Schoenberg, Andrew, and Lock[7,8] have determined the order of magnitude of Δ by analyzing the shape of the magnetization curves of small specimens. The indefiniteness of the theoretical premises concerning the structure of the intermediate state, and also, apparently, the marked effect of the lack of an equilibrium state in the sample, led to large dispersion of the data, which differed among themselves by as much as a factor of 3 or 4 or more. Most of the data obtained were still two times smaller than our data for the same temperature.

Faber[9] studied the dependence of Δ on temperature by measuring the rate of enlargement of nucleii of the superconducting phase in a supercooled specimen. The difficulty of carrying out this process, as the author noted, made it possible to estimate the absolute value of the magnitude of Δ only to within 50%, but the relative data concerning the temperature dependence appear to be more accurate (to within about 5%). These data are in good agreement with our measurements, since the ratio of our value of Δ to that given by Faber is equal to 1.65 over the entire temperature interval, with unsystematic deviations of less than 5% for individual points.

In Fig. 7 is also plotted a curve for Δ calculated according to the theory of Ginzburg and Landau[10] with account taken of the more accurate approximation to the relation between Δ and the penetration depth δ, obtained by Ginzburg.[11] The value of δ measured by Laurmann and Schoenberg[12] was used. The ratio of the theoretical value of Δ to the measured one is 1.5 near T_c and depends substantially on the temperature, decreasing to 1.2 at 2.165° K. This difference, although certainly not large, is nevertheless outside the limits for the probable experimental error for the measurements of Δ and δ (about 5 – 8% in the experiment of Laurmann and Shoenberg[12]). It should be noted that other measurements by the author[13] have given a value of δ approximately 8% smaller, and the recent work of Chambers[14] a value 16% smaller than that of Laurmann and Shoenberg.[12] Such a decrease in the value of δ would lead to a further increase of the discrepancy between theory and experiment, since Δ is approximately proportional to $1/\delta$.

The possible explanation of the disagreement as a result of the anisotropy of the superconductive properties of tin is not a very likely one, since the anisotropy of Δ according to preliminary data does not exceed 10%.[15] The anisotropy of δ in the estimates which have been made also is not sufficiently large. Nonetheless, the problem of the nature and magnitude of the anisotropy of Δ and δ is at present completely confused, and its solution is necessary for clarifying the source of the observed discrepancy and for directing further improvements of the theory.

I express my sincere thanks to P. L. Kapitza for his interest in the work and to A. K. Shal'nikov for a detailed discussion and much valuable advice.

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COULOMB EXCITATION OF SEPARATED TIN ISOTOPES

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Alpha particles accelerated to 14.5 Mev in a cyclotron were used to investigate the Coulomb excitation of nuclear levels of separated tin isotopes. A number of previously unknown states were excited, for which we measured the energy and the reduced probability of transition from the ground state. The lifetimes of the first excited states of even-even tin isotopes were calculated. The lifetimes lie in the range from 5×10^{-13} sec (Sn^{114}) to 16×10^{-13} sec (Sn^{124}).

1. INTRODUCTION

THE study of Coulomb excitation of rotational levels of nuclei in the mass region $A = 150 - 190$ has played an important part in checking and improving the concepts of the uniform model. At the present stage of development of the uniform model, it is of interest to study Coulomb excitation of levels in nuclei which are outside the rotational region. It is especially important to excite levels of even-even nuclei, where the internal structure does not affect the angular momentum of the nucleus. However, efforts to excite levels by Coulomb interaction have certainly not been successful for all nuclei outside the rotational region. Thus, attempts[1,2] at Coulomb excitation of levels in tin, using α particles with an energy E_α up to 7 Mev, gave no results.

The cross section for Coulomb excitation via electric quadrupole transition is given by the formula

$$\sigma = 2\mu (E - \Delta E) B(E2) f_2(\xi)/Z_2^2 \hbar^2 e^2, \qquad (1)$$

where

$$\xi = \frac{Z_1 Z_2 e^2 \sqrt{\mu}}{\hbar \sqrt{2}} \left(\frac{1}{\sqrt{E - \Delta E}} - \frac{1}{\sqrt{E}} \right) \approx \frac{Z_1 Z_2 e^2 \mu^{1/2}}{\hbar \sqrt{2}} \frac{\Delta E}{2E^{3/2}}, \qquad (2)$$

μ is the reduced mass of the bombarding particle, Z_1 the number of protons in its nucleus, E and $(E - \Delta E)$ are the energy of the nucleus before and after collision, Z_2 is the number of protons in the nucleus of the target atom, $B(E2)$ is the reduced probability for electric quadrupole transition of the nucleus from its ground state to the particular excited state, ΔE is the nuclear excitation energy, and $f_2(\xi)$ is the Coulomb excitation function, which for not too low values of ξ drops almost exponentially with increasing ξ. A graph of $f_2(\xi)$ is given in Ref. 3.

Considering that the values of ΔE for the low-lying levels of tin are large, and that the values of $B(E2)$ for excitation of vibrational or single-particle levels are small compared to the values for excitation of rotational states, it follows from formulas (1) and (2) that the cross section for excitation of levels in tin should be very small if α particles with E_α below 7 Mev are used. However, as our computations show, when E_α is increased from 7 to 13 Mev the thick target yield (with $\Delta E = 1.2$ Mev) increases by a factor of 200.

To excite nuclear levels in tin, we used α particles accelerated to 14.5 Mev in a cyclotron. Previously most experiments on Coulomb excitation have been done with particles accelerated in an electrostatic generator. Then the energy spread in the beam is very small and it is also easy to change the beam energy. However, raising the energy of particles accelerated in an electrostatic generator presents well-known difficulties so that up to now, in work on Coulomb excitation, particles with energies not exceeding 7 Mev have been used.

The energy spread of the particles in the extracted cyclotron beam does not exceed ± 2%. Our computations show that if, instead of a monochromatic beam with $E_\alpha = 10$ Mev, a beam with a Gaussian energy distribution with 2% half-width is used for Coulomb excitation, the error in the determination of $B(E2)$

does not exceed 1%. If, as an extreme case, we assume that the energies of all the particles in the beam differ from the nominal energy by 2%, the resultant error in the determination of B(E2) is 15% for $E_\alpha = 10$ Mev and 10% for $E_\alpha = 14.5$ Mev, if $\Delta E = 1.2$ Mev. Since the overall error in the determination of B(E2) in Coulomb excitation experiments, due to uncertainties in various other factors, is usually not less than 20%, the resultant error if we include the possible error associated with the energy spread is $\sqrt{20^2 + 15^2}$, i.e., 25%.

Increasing the value of E_α by a factor of 2 compared to the values used in other work on Coulomb excitation should lead to a large increase in the background of radiations which are emitted in reactions which occur via the formation of a compound nucleus. Preliminary estimates showed that, for $E \approx (0.6 - 0.8)E_B$, the cross section for compound nucleus formation is less than or comparable with the cross section for Coulomb excitation for nuclei in the middle of the periodic table. In studying Coulomb excitation, the most important contamination comes from inelastic scattering (α, α') which occurs via compound nucleus formation, since γ rays of the same energy are then emitted as in the (α, α') process which occurs via Coulomb excitation. However, the relative probability for decay of the compound nucleus via the (α, α') channel is much less than the probability for decay into (α, n), (α, p), and (α, α) channels.

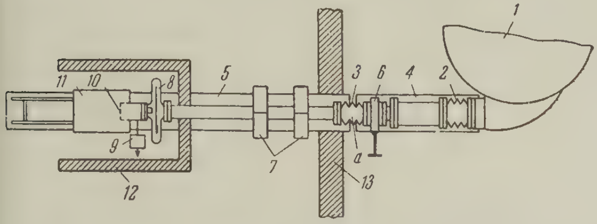

FIG. 1. Arrangement of apparatus (top view): 1 — cyclotron chamber; 2 and 3 — syphon bellows; 4 and 5 — beams, linked by a universal joint which makes possible rotation of beam 5 around point a in any direction and thus simplifies setting the system so that the particle beam strikes the center of the target; 6 — vacuum gate, which is closed for changing targets; 7 — magnetic lenses; 8 — glass section for observing scintillation screen to enable focusing the beam; 9 — current integrator input stage; 10 — target; 11 — FEU-S photomultiplier in iron shield; 12 — lead shield against γ radiation from cyclotron; 13 — shield against neutrons from cyclotron. It consists of paraffin blocks for moderating the neutrons and cadmium sheets to absorb them.

We should mention that the theoretical energy dependence of the cross section for Coulomb excitation has been verified in many experiments over a wide range of energies of the bombarding particles. Consequently, experimental determination of the excitation function enables us to decide whether a particular line is the result of Coulomb excitation or of compound nucleus formation.

2. EXPERIMENTAL METHOD

A beam of α particles with energies up to 14.5 Mev was obtained at the cyclotron of the Physico-Technical Institute of the Academy of Sciences. Using a standard deflector, the beam of α particles was brought out into an evacuated aluminum tube. A series of lead diaphragms were arranged inside the tube to prevent the particles from striking the tube walls. The beam was focused on the target by a system of two magnetic quadrupole lenses.* The target was placed on the bottom of an insulated metallic cup which served as a Faraday cup. The amount of charge carried by the α particles bombarding the target was measured using an electronic integrator, which was calibrated before and after each experiment. To eliminate errors in beam current measurement due to secondary electron emission, a ring was placed in front of the Faraday cup and was kept at 90 volts negative with respect to it. The γ radiation emitted in de-excitation of the excited levels was detected with a scintillation spectrometer. The FEU-S photomultiplier, the NaI(Tl) crystal (diameter 29 mm and $\ell_k = 14$ mm), and a cathode follower were kept inside an effective magnetic shield consisting of three concentric iron cylinders. To avoid errors resulting from the anisotropy in the angular distribution of the γ quanta emitted after Coulomb excitation, it is desirable to achieve an arrangement of the apparatus corresponding to a solid angle $\sim 2\pi$. In our experiments the front surface of the crystal was at a distance of 4.35 mm from the target. Under conditions of near geometry, small displacements of the crystal relative to the target give a sizeable change of solid angle, i.e., of counting efficiency. We therefore took special measures to assure a unique geometry: the photomultiplier and crystal were pressed against the bottom of the Faraday cup by means of a spring, the bottom of

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*The parameters of the magnetic lenses and some additional information concerning procedure are given in other papers.[4,5]

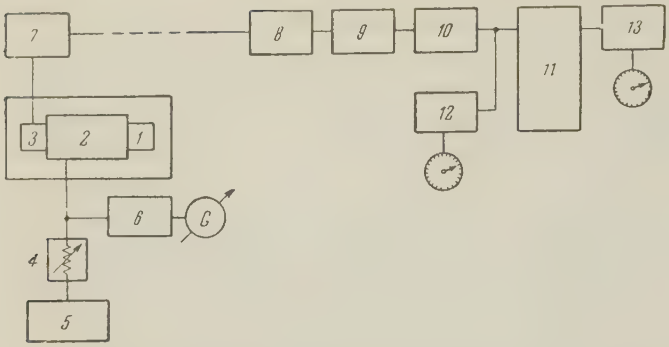

FIG. 2. Block diagram of the γ-spectrometer: 1 — scintillator; 2 — photomultiplier; 3 — cathode follower; 4 — regulating resistor for maintaining constant voltage on the photomultiplier; 5 — dry battery; 6 — potentiometer circuit for voltage measurement; 7 — heavy-duty cathode follower, feeding into a 50 meter line; 8 — linear amplifier; 9 — pulse stretcher; 10 — window amplifier; 11 — 50-channel pulse-amplitude analyzer; 12 — scaling circuit for N_0 counts; 13 — scaling circuit for N_B counts.

the copper Faraday cup was made thick enough so that it would not bend under atmospheric pressure; finally, a special arrangement assured good contact between the target backing and the bottom of the Faraday cup.

Between the target and the front surface there were the following layers of material, which partially absorbed the γ rays from the target: 400μ Cu, 1.3 mm Al, 100μ mica, 1 mm MgO, 50μ Pb, and 1.5 mm air.

A diagram of the apparatus is shown in Fig. 1. The targets were separated isotopes of tin in the form of metallic samples. The target backing was a 50μ thick layer of lead pressed into an aluminum cap 1 mm high, 28 mm in diameter, with a base 300μ in thickness. The metallic tin sample was placed at the center of the target and was pressed onto the backing along with a lead disc which was placed on top of it and which had an opening at its center 8.2 mm in diameter. The fraction of the flux striking the tin target was determined in separate experiments in which we compared the yield of γ radiation emitted after Coulomb excitation of tantalum when the target was a tantalum disc 8.2 mm in diameter placed at the center, and when the tantalum disc had a diameter equal to the inner diameter of the aluminum cup. In experiments on Coulomb excitation of tellurium, a water suspension of metallic tellurium powder was deposited in droplets on the central portion of the backing. After evaporation of the water, the tellurium powder was pressed in the same way as for the tin samples. In all cases, the target thickness was sufficient to slow down the α particles to an energy at which the Coulomb excitation yield was insignificant. At first we had intended, in getting the Coulomb excitation function, to vary the α-particle energy by placing aluminum foils of various thicknesses along the beam path. However, it turned out that even when the foils were located 60 cm from the target, the background rose as a result of α-particle reactions in Al by an order of magnitude compared with its level when the foils were absent. It was not possible to reduce the background by moving the foils further from the target, since the intensity of the α-particle beam incident on the target was diminished because of Rutherford scattering. For this same reason (increase in Rutherford scattering with increasing Z_2), we could not reduce the background from nuclear reactions by replacing the aluminum by an element with higher atomic number. We succeeded in using the foils only when the yield from Coulomb excitation was large compared to the background (for example, in experiments on Coulomb excitation of Ta and Ag which were done to calibrate the apparatus). We therefore avoided the use of the foils in getting the Coulomb excitation function, and changed the α-particle energy in steps, by readjusting the cyclotron. Altogether, we used four fixed values of the α-particle energy: 10.15, 11.45, 13.1, and 14.5 Mev.

A block diagram of the γ-spectrometer is shown in Fig. 2. A blocking arrangement was provided in the circuit of the 50-channel analyzer, to cut off the analyzer input from the window-amplifier output for a period of 10μ sec after the acceptance of a pulse by the analyzer. The window-amplifier circuit is linear, and sends pulses to the analyzer input having amplitudes in the range from V_1 to V_2. The values for the lower and upper limits of the window-amplifier output could be varied in the range from 5 to 80 volts. The purpose of the window amplifier is to reduce the overall load on the analyzer. To calculate the absolute values of the cross section, the number of counts in the channels must be multiplied by the ratio N_0/N_B, where N_0 is the total number of pulses passed by the window amplifier, and N_B is the number of pulses accepted for analysis by the amplitude analyzer.

The energy calibration of the γ-spectrometer was done using γ rays from Hg^{203} (279.5 kev); Cs^{137} (661 kev), and Zn^{65} (1120kev).

3. HANDLING OF RESULTS

The theoretical value of the total yield Y, per bombarding particle, for excitation of a state with energy ΔE in bombardment of a thick target, is

$$Y = \int \sigma N_1 dx = \frac{2\mu N_A}{A_2 Z_2^2 \hbar^2} \frac{B(E2)}{e^2} \int\limits_{\Delta E}^{E_{\max}} \frac{(E - \Delta E) f_2(\xi) dE}{dE / d\rho x} . \qquad (3)$$

N_1 is the number of nuclei of the isotope per cc, N_A is Avogadro's number, A_2 is the mass number of the isotope, $dE/d\rho x$ is the stopping power of the target material; $E_{\max}$ is the collision energy corresponding to the initial energy of the α particles bombarding the target.

In the experiment the yield is found by taking the spectrum of Coulomb excitation and determining the number of γ quanta per microcoulomb, S_{ph}, recorded under the full energy peak. The full energy peak corresponds to complete absorption of the energy of the γ quantum in the crystal, and

$$Y = 3.2 \cdot 10^{-13} S_{ph} (1 + \alpha_t) / \eta \epsilon_{ph} \omega A_\gamma . \qquad (4)$$

In formula (4), α_t is the total internal conversion coefficient [the factor $(1 + \alpha_t)$ takes account of the possibility of de-excitation via conversion], ϵ_{ph} is the fraction of γ quanta recorded under the full energy peak, relative to the total number of γ quanta striking the crystal; ω is the solid angle in units of 4π; A_γ is a factor which takes account of the absorption of the γ rays in the target and in the materials which are located between the target and the crystal; η is the percentage content of the particular isotope in the target. The value of $dE/d\rho x$ was calculated from the formula

$$-dE / d\rho x = (4\pi e^4 Z_1^2 Z_2 N_A / m v^2 A_2) L, \qquad (5)$$

where m is the electron mass and v the velocity of the incident particle. To compute L, we used the semi-empirical formula:[6]

$$L = 1.55 v \hbar / e^2 Z_2^{1/2}. \qquad (6)$$

Thus,

$$-dE / d\rho x = 4.38 \pi e^2 \hbar N_A Z_1^2 Z_2^{1/2} \mu^{1/2} / m A_2 E^{1/2} = K E^{-1/2}. \qquad (7)$$

To check whether formula (7) could be used reliably for determining $dE/d\rho x$, we used the formula for Cu, Sn, Pb, for four values of E_α in the range from 15 to 10 Mev, and compared the results with data of Rybakov[7] and Segre.[8] The agreement was good. The difference did not exceed 5% for Cu and Sn, and was less than 10% for Pb.

The final expression for determining B(E2) when α particles are used for Coulomb excitation is gotten from formulas (3) – (6) and has the form

$$\frac{B(E2)}{e^2} = 6.254 \cdot 10^{-10} \frac{Z_2^{5/2}}{\mu^{1/2} \eta} \left\{ S_{ph} (1 + \alpha_t) / \epsilon_{ph} \omega A_\gamma \int\limits_{\Delta E}^{E_{\max}} (E - \Delta E) E^{1/2} f_2(\xi) dE \right\} , \qquad (8)$$

where E and ΔE are in Mev, μ is in mass units and $B(E2)/e^2$ in units of 10^{-48} cm^4. The values of the integral were computed by graphical integration.

The internal conversion coefficients are very small (less than 1%) for the high-energy levels excited in tin, and α_t was set equal to zero in the computations.

The estimate of the overall efficiency $\epsilon_{ph} \omega A_\gamma$ of the scintillation spectrometer was done as follows: the values of ϵ_{ph}, ω, and A_γ were calculated individually as a function of the γ-ray energy E_γ, and a graph was made of the product of these quantities as a function of E_γ in the range of E_γ from 0 to 1.4 Mev. ϵ_{ph} was determined from the relation

$$\epsilon_{ph} = (1 - \exp\{-\mu_t l\}) p^* , \qquad (9)$$

where μ_t is the total absorption coefficient for the γ rays in the material of the crystal, ℓ is the height of the crystal, and p^* is the fraction of the γ quanta recorded under the full energy peak relative to the total number recorded over the whole spectrum. The magnitude of p^* depends on the energy of γ quanta, the dimensions of the crystal and the geometry for recording the γ radiation (the shape of the source, its position and distance from the crystal). To calculate ϵ_{ph}, we used the values of μ_t and p^* given in Ref. 9. The solid angle was computed from the formula

$$\omega = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + r^2/h'^2}}\right), \tag{10}$$

where r is the radius of the crystal, $h' = h + \ell'$, h is the distance from the target to the front face of the crystal, ℓ' is the distance to the plane in the crystal such that equal numbers of γ rays are absorbed on each side of it, i.e.,

$$1 - \exp\{-\mu_t l'\} = \frac{1}{2}(1 - \exp\{-\mu_t l\}); \tag{11}$$

$$l' = -\frac{1}{\mu_t} \ln \frac{1}{2}(1 + \exp\{-\mu_t l\}). \tag{12}$$

Values of γ-ray absorption coefficients used in calculating A_γ were taken from Ref. 10.

To check the computations, we determined the counting efficiency at the full energy peak for γ quanta with $E_\gamma = 1.33$ Mev (Co^{60}) and 0.661 Mev (Cs^{137}). For this purpose we prepared thin sources of Co^{60} and Cs^{137}, having the same shape as the targets used in the experiments on Coulomb excitation. The intensity of these sources was determined using a 4π-geometry β-counter.† The gamma spectra of Co^{60} and Cs^{137} were taken in the same apparatus, keeping the same geometry and with the same absorbing materials present as in the experiments on Coulomb excitation. The values of total efficiency found by experiment exceeded the calculated values by a factor of 1.30 for $E_\gamma = 661$ kev and a factor of 1.27 for $E_\gamma = 1.33$ Mev. In computing values of B(E2) from the Coulomb excitation experiments, we used our computed values of the total efficiency multiplied by 1.28.

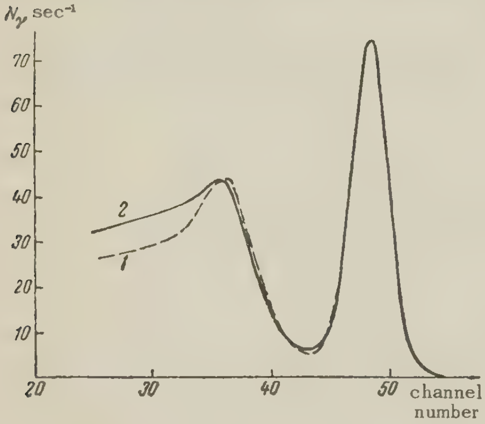

FIG. 3. 1 — theoretical line shape for $E_\gamma = 1.12$ Mev, computed from data of Ref. 9 using the measured values for the energy resolution of the spectrometer; 2 — experimental spectrum of Zn^{65} γ radiation ($E_\gamma =$ 1.12 Mev).

The difference between the computed and measured values of the overall efficiency is apparently explained by the fact that our experiments were carried out under conditions of "near geometry," whereas the values of μ_t and p^* are given[9] for the case where the beam of γ rays is incident parallel to the axis of the crystal, i.e., for "far geometry." The problem of the dependence of the efficiency of recording γ rays on the distance of the source from the crystal is investigated in Ref. 12. Using the data of this paper, we recomputed the values of the overall efficiency for the case of near geometry. The computed values corrected in this fashion were in good agreement with the data obtained experimentally.

The magnitude of S_{ph} was determined from the area under the full energy peak. The area of the peak, the position of the maximum, the height of the maximum and the peak width all depend on how one draws the background curve. To determine the value of S_{ph} more accurately, we used the theoretical line shape constructed from data given in Ref. 9. By repeated trials with slight changes in the position of the maximum, its height and the half width of the peak, we tried to subtract the theoretical spectrum from the observed spectrum and obtain a smooth background curve which joined continuously to the experimentally observed spectrum of the background to the right of the peak. To check the computations of the theoretical line shape, the shape of the 1.12 Mev line from Zn^{65} was measured with the scintillation spectrometer. The results are shown in Fig. 3. The figure shows that the experimental and computed spectra are in good agreement. The somewhat higher intensity of the experimental spectrum in the low-energy region is explained by the contribution from γ quanta scattered in the iron cylinders which surrounded the photomultiplier and crystal.

In determining S_{ph} we also had to take account of the presence of other isotopes in the sample. Using the known values for the content of isotopic impurities, the energies of levels excited in them, and the values of B(E2) for excitation of these levels, we could calculate the theoretical spectrum for the lines

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† The counter was prepared on the basis of data presented in Ref. 11.

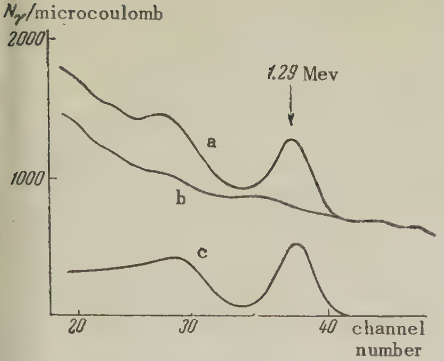

FIG. 4

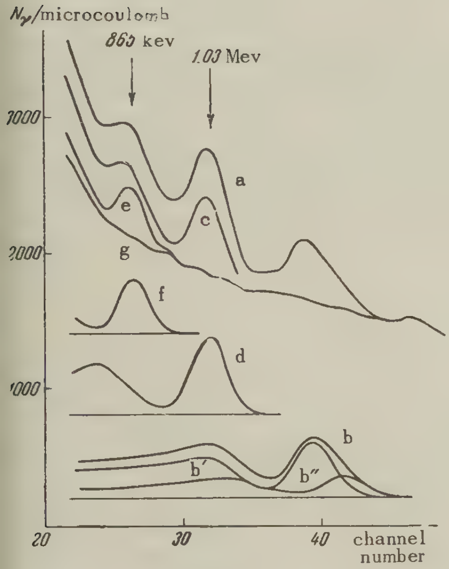

FIG. 5

FIG. 4. Gamma spectrum emitted after bombardment of Sn^{116} with 13.1 Mev α particles: a — experimental spectrum; b — theoretical line shape for γ quanta with $E_\gamma = 1.29$ Mev; c — background curve after subtracting theoretical line shape from experimental spectrum.

FIG. 5. Gamma spectrum emitted after bombardment of Sn^{117} with 13.1 Mev α particles: a — experimental spectrum; b′ and b″ — theoretical line shapes for $E_\gamma = 1.24$ and 1.31 Mev; b — curve obtained by adding ordinates of curves b′ and b″; c — spectrum after subtracting curve b; d — theoretical line shape for $E_\gamma = 1.03$ Mev. e — spectrum after subtracting curve d; f — theoretical shape for $E_\gamma = 865$ kev; g — background curve.

emitted in the de-excitation of the states excited in the impurity isotope. This spectrum was subtracted from the spectrum measured in the experiment.

Figures 4 and 5 show γ spectra emitted after Coulomb excitation of Sn^{116} and Sn^{117}.

In calculating values of B(E2) for lines excited in even-even isotopes, the anisotropy of the angular distribution of the γ rays emitted after Coulomb excitation was taken into account. The values of B(E2) obtained from formula (8) omitting the effects of anisotropy were approximately 5% greater than the results when the anisotropy was included.

We can name various causes which affect the accuracy in the determination of B(E2). One is the inaccuracy in determining the efficiency of the crystal for counting γ quanta under the full energy peak, others are the inaccuracy in determining $dE/d\rho x$, errors in the integrator, errors caused by the lack of definiteness in drawing the background curve under the peak, errors due to the energy spread of the α particles in the beam, and finally errors due to counting statistics. According to our estimates, none of these errors exceed 10% and the overall error in determining B(E2) should be 25%. As a check, we took the γ-ray spectra emitted after Coulomb excitation of Ta, Ag, Mo, and Te^{130}. The deviation of the average values of B(E2) for states excited in these nuclei in our experiments from data in the literature did not exceed 25%.

4. RESULTS OF MEASUREMENTS AND DISCUSSION

1. Even Isotopes of Tin

The data obtained for Coulomb excitation of even isotopes of tin are given in the table. ΔE is the energy of the first excited state according to our data, while ΔE^* is the energy obtained from studies of β-decay or inelastic scattering of neutrons.

The values of ΔE found by us fill the gaps in the classification of energy values of first excited states of even-even isotopes of tin. Levels not previously known were found in Sn^{112}, Sn^{118}, and Sn^{124}. For Sn^{120} and Sn^{116}, the data of various workers gave different values of ΔE^*. In these cases, the data in the table enable us to make the correct choice. The observed trend in ΔE with changing atomic number is possibly to be explained by the occurrence of a closed subshell at N = 64.

Isotope	ΔE, Mev	ΔE^*, Mev	$\frac{B(E2)}{e^2}\cdot 10^{48}$, cm^4	$\tau 10^{13}$, sec	F	$B_2/B_2{}^*$	C_2, Mev
Sn^{112}	1.26	—	0,18	7.2	11.1	16.3	280
Sn^{114}	1.30	1.3 [13]	0.20	5.5	12.0	14.2	266
Sn^{116}	1.29	1.27 [14] 1.3 [15]	0,19	6.0	11.2	14.9	285
Sn^{118}	1.22	—	0.19	8.0	11.0	15.7	276
Sn^{120}	1.18	1.18 [16] 1.3 [17]	0.17	10.5	9.6	18.0	304
Sn^{122}	1.15	1.14 [18]	0.15	13.5	8.2	20.8	344
Sn^{124}	1.13	—	0.14	15.9	7.6	22.6	370

The excitation function was taken for each of the lines listed in the table, and for the values of E_α mentioned earlier. We observed no systematic change in B(E2) with changing E_α. The values of B(E2) found for different values of E_α remained constant within the limits of error of the experiment. This is evidence for the Coulomb character of the excitation. The table gives the average values of $B(E2)/e^2$ found from a series of measurements including measurements for different values of E_α. The average value of the rms error in determining B(E2) for the various isotopes was 20%. The largest rms error was 35% for Sn^{114}, and the smallest 15% for Sn^{124}. Levels with large values of ΔE were excited in our work. The accuracy in estimating the efficiency of the crystal depends on E_γ. It is therefore especially interesting to compare values of B(E2) for excitation of states with large values of ΔE. For this purpose we studied the Coulomb excitation of the state with $\Delta E = 857$ kev ($\Delta E = 850$ kev according to Ref. 1) in Te^{130}. According to our data the value of $B(E2)/e^2$ is 0.24×10^{-48} cm^4, while Ref. 1 gave 0.26×10^{-48} cm^4.

From the table it follows that the values of B(E2) differ very little for all the even-even tin isotopes. The differences are within experimental error. The average value of $B(E2)/e^2$ is approximately 0.17×10^{-48} cm^4

From the values of B(E2) and ΔE determined in the experiment we can determine various other quantities characterizing the nucleus, in particular we can find the lifetime τ of the first excited states of the even-even isotopes. Since the spin of these states is 2^+, their de-excitation can occur only via an E2 transition to the 0^+ ground state. The conversion coefficients for these transitions in tin are very small, so that in determining τ we can use the formula given in Ref. 19 for the lifetime for a radiative transition with emission of a photon of multipolarity λ and energy ΔE:

$$\frac{1}{\tau_\gamma(\lambda)} = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2}\,\frac{1}{\hbar}\left(\frac{\Delta E}{\hbar c}\right)^{2\lambda+1} B(\lambda)\downarrow . \tag{13}$$

It follows that, for an E2 transition ($\lambda = 2$),

$$\frac{1}{\tau_\gamma(E2)} = \frac{4\pi}{75}\left(\frac{\Delta E}{\hbar c}\right)^5 \frac{1}{\hbar} B(E2)\downarrow , \tag{14}$$

where $B(E2)\downarrow$ is the reduced probability of transition from the excited state to the ground state. According to the principle of detailed balancing, the values of $B(E2)\downarrow$ and $B(E2)\uparrow$ are related by

$$(2I_f + 1)\,B(E2)\downarrow = (2I_0 + 1)\,B(E2)\uparrow . \tag{15}$$

Since in our case $I_0 = 0$, $I_f = 2$,

$$B(E2)\downarrow = {}^1\!/_5 B(E2)\uparrow . \tag{16}$$

Values of τ calculated using formulas (14) and (16) are given in the table. The values of τ determined in the experiment lie in the range from 5.5 to 15.9×10^{-13} sec. We note that satisfactory measurements of lifetimes using delayed coincidences have so far been limited to values of τ greater than 10^{-11} sec.

The quantity F given in the table is the ratio of $B(E2)\uparrow$ to the quantity $B(E2)_{s.p.}\uparrow$ calculated on the basis of the single-particle model:

$$B(E2)_{\text{s.p.}}\uparrow = 5\left(\frac{1}{4\pi}\right)\left(\frac{3}{5}R_0^2\right)^2 .$$

In the computations we used $R_0 \approx 1.2\,A^{1/3} \times 10^{-13}$ cm. It follows from the table that the value of F for even-even tin isotopes is approximately equal to 10. This is an indication of the collective nature of the excited states.

The last two columns of the table give the values of the parameters B_2/B_2^* and C_2 which can be obtained from the experimental values of B(E2) and ΔE if we assume that the first excited states of the even-even tin isotopes are vibrational levels. In the liquid drop model, for vibrations with small amplitude the expression for the energy can be expanded in a series in the amplitudes $\alpha_{\lambda\mu}$ of oscillation of the electric multipole moment and their rates of change, $\dot{\alpha}_{\lambda\mu}$, so that in first approximation

$$H_{\text{vib}} = \sum\left\{\frac{1}{2}B_\lambda(\dot{\alpha}_{\lambda\mu})^2 + \frac{1}{2}C_\lambda(\alpha_{\lambda\mu})^2\right\}.$$

B_λ is a parameter characterizing the mass transfer associated with the vibrations, and C_λ is the surface tension parameter. Since the surface deformation for $\lambda = 1$ is simply a displacement of the mass, the lowest frequencies of collective oscillation correspond to quadrupole vibrations ($\lambda = 2$). Reference 19

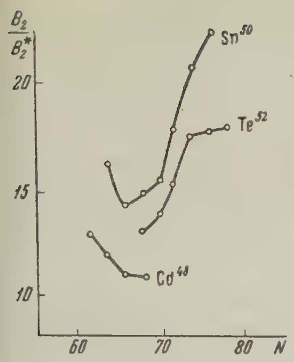

FIG. 6

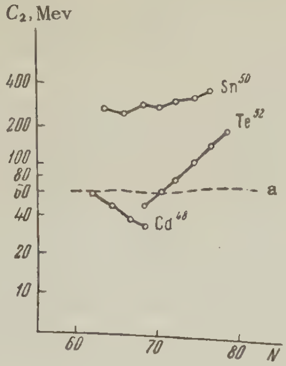

FIG. 7

FIG. 6. Dependence of B_2/B_2^* on neutron number N for even-even tin isotopes. For comparison with our data on tin, we give the data of Ref. 1 for Cd and Te.

FIG. 7. Dependence of C_2 on neutron number for even-even tin isotopes. For comparison with our data on tin, we give data from Ref. 1 for Cd and Te; curve a shows the values of C_2 according to the liquid drop model.

gives formulas for determining B_2 and C_2 from experimental values of $B(E2)$ and ΔE:

$$B(E2) = 5\left(\frac{3}{4\pi} ZeR_0^2\right)^2 \frac{\hbar}{2(B_2C_2)^{1/2}}; \quad \Delta E = \hbar\sqrt{C_2/B_2}.$$

The table gives the ratios B_2/B_2^*, where B_2^* is the value of the mass transfer parameter for the case of surface oscillations of an irrotational incompressible liquid drop. B_2^* is computed from the formula

$$B_2^* = \frac{3}{8\pi} AMR_0^2,$$

where A is the atomic number and M is the nucleon mass.

In Figs. 6 and 7, our values for B_2/B_2^* and C_2 for the even-even isotopes of tin ($Z = 50$) are compared with values of B_2/B_2^* and C_2 for Cd ($Z = 48$) and Te ($Z = 52$) given in Ref. 1. The dashed curve in Fig. 7 shows the value of C_2 obtained from the liquid drop model. According to the figures, the values of B_2/B_2^* are much greater than unity. The values of C_2 are approximately 5 times as great as the value of C_2 from the liquid drop model. It is interesting to note that, unlike cadmium and tellurium (which have almost the same Z as tin), the values of C_2 for the tin isotopes change very little with changing neutron number.

The vibrational states have various characteristic features.[19] The energy of the second excited state is usually 2 — 2.5 times the energy of the first excited state. If the spin of the second level is 2^+, the probability of an M1 transition to the first excited state with spin 2^+ is greatly reduced. In addition, the reduced probability for E2 transition from the second level to the ground state, which is forbidden for harmonic vibrations, is several orders of magnitude less than the transition probability to the first state. Finally, the cross section for Coulomb excitation of vibrational levels is approximately an order of magnitude greater than that for single-particle transitions, i.e., $F > 1$. The fact that, according to our data, $F \approx 10$, does not exclude the possibility that the first excited levels of tin are single-particle levels, since collective interactions can increase the probability of a single-particle transition. To arrive at a definite conclusion concerning the nature of the first excited levels of the even-even tin isotopes, it would be important to excite the second levels. This problem is, however, much more complicated, both because of kinematic considerations as well as because the reduced transition probability from the ground state to the second state is much less than that to the first excited state.

2. Odd Isotopes of Tin

Our spectra of the Coulomb excitation of tin isotopes always contained two intense peaks, one peak at $E = 75$ kev, due to the characteristic radiation from lead induced by the α particles, and a peak with $E = 511$ kev resulting from annihilation of positrons emitted from radioactive elements formed in (α, n) reactions. We therefore investigated only the regions of the spectrum with energies from 0.1 to 0.4 Mev and energies above 0.75 Mev. No lines were found in the energy region from 0.1 to 0.4 Mev which could be attributed to Coulomb excitation. In the bombardment of Sn^{115}, we also found no lines in the region from 0.75 to 1.75 Mev. In bombardment of Sn^{119}, we found a line with $E = 0.907$ Mev $[B(E2)/e^2 = 0.11 \times 10^{-48}$ cm$^4]$. In bombarding Sn^{117}, we found two lines, with $E = 0.865$ Mev $[B(E2)/e^2 = 0.025 \times 10^{-48}$ cm$^4]$ and $E = 1.03$ Mev $[B(E2)/e^2 = 0.09 \times 10^{-48}$ cm$^4]$. The values of $B(E2)$ were calculated under the assumption that the values of the conversion coefficients for these transitions are small, and that the energies of the observed lines are equal to the excitation energies of the corresponding states. In other words, we assumed that these lines are not emitted as the result of a cascade from higher levels.

In the bombardment of Sn^{117}, we also found a composite peak, which we resolved into lines with $E =$

1.24 and 1.31 Mev. Study of the excitation function led us to the conclusion that these lines should be assigned to nuclear reactions which proceed via compound-nucleus formation.

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 6 JUNE, 1958

ON THE THERMODYNAMICAL THEORY OF RELAXATION PHENOMENA IN SYSTEMS WITH ADDITIONAL PARAMETERS

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A thermodynamical theory of relaxation phenomena connected with the relaxation of an additional internal parameter of the system is developed on the basis of the method of Mandel'shtam and Leontovich. The relaxation times and the relation between them are established. Expressions are obtained for the dynamic derivatives of a system undergoing a periodic perturbation. A relation is found between the relaxation times and discontinuities in the derivatives near the Curie point. The results are extended to an arbitrary number of additional internal parameters.

THE term "additional internal parameter" is applied to a quantity characterizing some internal property of a system which, when the system is in equilibrium, appears to be a function of state. A delay on the par of the additional parameters in responding to external influences on the system leads, for example, to relaxation absorption of sound in liquids[1,2] or to internal friction in solids.[3] Below we develop further the method proposed by Mandel'shtam and Leontovich[1] for explaining acoustical absorption in liquids.

We consider a system characterized in its equilibrium state by a temperature T, a generalized force A (pressure, stress), and a coordinate a conjugate to it (volume, strain), connected by an equation of state; and some additional internal parameter $\eta = \eta(T, a)$. This parameter may, for example, be the degree of long- or short-range order, the spontaneous magnetization, the spontaneous electric polarization, the antiferromagnetic order, etc.

When the state of the system changes in time, the parameter η takes non-equilibrium values $\eta \neq \eta(T, a)$; consequently, the system as a whole will pass through non-equilibrium states, in which it is described by the free energy $F = F(T, a, \eta)$. During this time we have $F_\eta(T, a, \eta) \neq 0$, where the subscript denotes, as usual, differentiation with respect to the corresponding parameter. When the system is in its equilibrium state,

$$F_\eta(T, a, \eta) = 0, \; F_{\eta\eta} > 0. \tag{1}$$

According to Leontovich,[4] the entropy change of a non-equilibrium state is equal to

$$T\,dS = dU + A\,da - F_\eta\,d\eta,$$

where $U(T, a, \eta)$ is the internal energy. Here the last term on the right determines the non-equilibrium part of the entropy change; we thus obtain for the rate of entropy production

$$T\Delta\dot{S} = -F_\eta\dot{\eta},$$

From this we have, in the approximation of the thermodynamics of irreversible processes,

$$\dot{\eta} = -LF_\eta(T, a, \eta). \tag{2}$$

Since $\Delta\dot{S} > 0$, $L > 0$.

Expanding F_η in a series about the equilibrium state of the system and taking note of (1), we obtain from (2), in the linear approximation, an equation describing the time rate of change of the parameter η:

$$\tau^{aT}\dot{\eta} + \Delta\eta = \left(\frac{\partial\eta}{\partial a}\right)_T \Delta a + \left(\frac{\partial\eta}{\partial T}\right)_a \Delta T. \tag{3}$$

where $\tau^{aT} = (LF_{\eta\eta})^{-1}$ is the relaxation time of the parameter η for constant a and T.

In order to go over to other variables in (3), we set $A = -F_a$ and the entropy $S = -F_T$, so that to a linear approximation near the equilibrium state of the system

$$\Delta A = -F_{aa}\Delta a - F_{aT}\Delta T - F_{a\eta}\Delta\eta; \tag{4}$$

$$\Delta S = -F_{TT}\Delta T - F_{aT}\,\Delta a - F_{T\eta}\Delta\eta. \tag{5}$$

The derivatives F_{aa}, F_{aT}, and F_{TT}, taken in the equilibrium state, can be expressed[1] in terms of the derivatives of the equilibrium state and $F_{\eta\eta}$:

$$F_{aa} = -\left(\frac{\partial A}{\partial a}\right)_T + \left(\frac{\partial\eta}{\partial a}\right)_T^2 F_{\eta\eta}; \quad F_{aT} = -\left(\frac{\partial A}{\partial T}\right)_a + \left(\frac{\partial\eta}{\partial a}\right)_T\left(\frac{\partial\eta}{\partial T}\right)_a F_{\eta\eta}; \quad TF_{TT} = -C_a + T\left(\frac{\partial\eta}{\partial T}\right)_a^2 F_{\eta\eta}, \tag{6}$$

where C_a is the specific heat of the system at constant a.

We have yet to explain the significance of the quantities entering into (6). If we exclude from our system the subsystem of the degrees of freedom responsible for the appearance of the properties characterized by the parameter η, i.e., if we set in (6) $(\partial\eta/\partial a)_T \equiv (\partial\eta/\partial T)_a \equiv 0$, then it becomes clear that, in the system remaining after the exclusion of the subsystem, $-F_{aa}$, $-F_{aT}$, and $-TF_{TT}$ represent the derivatives $(\partial A/\partial a)'_T$, $(\partial A/\partial T)'_a$, and the specific heat C'_a. Consequently, in turn, the quantities

$$\left[\left(\frac{\partial A}{\partial a}\right)_T\right] = \left(\frac{\partial\eta}{\partial a}\right)_T^2 F_{\eta\eta}; \quad \left[\left(\frac{\partial A}{\partial T}\right)_a\right] = \left(\frac{\partial\eta}{\partial a}\right)_T\left(\frac{\partial\eta}{\partial T}\right)_a F_{\eta\eta}; \quad [C_a] = T\left(\frac{\partial\eta}{\partial T}\right)_a^2 F_{\eta\eta} \tag{7}$$

are the respective contributions of the subsystem to $(\partial A/\partial a)_T$, $(\partial A/\partial T)_a$, and to the specific heat C_a of the whole system. In particular, if a second-order phase transition connected with the existence of the parameter η, is possible for our system, then at the Curie point $[(\partial A/\partial a)_T]$, $[(\partial A/\partial T)_a]$, and $[C_a]$ will be equal to the discontinuities of the corresponding quantities.

Using Eqs. (4) and (5) to transform (3) successively to pairs of variables other than a and T, we can show that the time rate of change of η, for an arbitrary choice of variables, has the form

$$\tau^{xy}\dot{\eta} + \Delta\eta = (\partial\eta/\partial x)_y\,\Delta x + (\partial\eta/\partial y)_x\Delta y, \tag{8}$$

where x and y are any two of the variables a, A, T, and S. Here the relaxation times τ^{xy} of the parameter η, for constant x and y, are related to each other by

$$\tau^{aT} = \tau^{AT}\frac{(\partial A/\partial a)_T}{(\partial A/\partial a)_T - [(\partial A/\partial a)_T]} = \tau^{Aa}\frac{(\partial A/\partial T)_a}{(\partial A/\partial T)_a - [(\partial A/\partial T)_a]} = \tau^{TS}\frac{(\partial A/\partial T)_a}{(\partial A/\partial T)_a - [(\partial A/\partial T)_a]} = \tau^{aS}\frac{C_a}{C_a - [C_a]}$$

$$= \tau^{AS}\frac{(\partial A/\partial a)_T C_A}{\{(\partial A/\partial a)_T - [(\partial A/\partial a)_T]\}(C_A - [C_A])}. \tag{9}$$

In deriving (9), use has been made of the equations in (6) and the notation of (7).

For a periodic perturbation of the system at a frequency ω, we can find the dynamic derivatives $(\partial y/\partial x)_{a,\omega}$, where x, y, z are any three of the variables a, A, T, or S. Eliminating $\Delta\eta$ from two equations of type (8), expressed in terms of the variables x, z and y, z, we obtain

$$\left(\frac{\partial y}{\partial x}\right)_{z,\omega} = \left(\frac{\partial y}{\partial x}\right)_z \frac{1 + i\omega\tau^{yz}}{1 + i\omega\tau^{xz}}. \tag{10}$$

From this it follows that

$$\left(\frac{\partial y}{\partial x}\right)_{z,\,\omega=\infty} : \left(\frac{\partial y}{\partial x}\right)_z = \tau^{yz} : \tau^{xz}. \tag{11}$$

For the variables P (pressure), V (volume), T, and S, the equations obtained by Davis and Lamb by another method[6] follow from (11) immediately.

From (8) it follows that

$$(\partial\eta/\partial x)_{y,\,\omega=\infty} \equiv (\partial\eta/\partial x)_{y,\,\omega=\infty} \equiv 0,$$

so that, for $\omega \to \infty$, the subsystem of the degrees of freedom responsible for the appearance of the system properties characterized by the parameter η take no part in the change of the state of the system; it is isolated from the remaining degrees of freedom of the whole system. Therefore the quantities $(\partial y/\partial x)_{z,\,\omega=\infty}$ are just the derivatives $(\partial y/\partial x)'_z$ relative to the system which remains after the subsystem has been excluded, and the quantity $[(\partial y/\partial x)_z] = (\partial y/\partial x)_z - (\partial y/\partial x)_{z,\,\omega=\infty}$ determines the contribution of the subsystem to the derivative $(\partial y/\partial x)_z$ for the total system. On this basis we obtain from (11)

$$\tau^{yz}/\tau^{xz} = 1 - \left[\left(\frac{\partial y}{\partial x}\right)_z\right] \Big/ \left(\frac{\partial y}{\partial x}\right)_z. \tag{12}$$

At the Curie point, $[(\partial y/\partial x)_z] = \Delta(\partial y/\partial x)_z$; consequently, one can write, near enough to the Curie point

$$\frac{\tau^{yz}}{\tau^{xz}} = 1 - \Delta\left(\frac{\partial y}{\partial x}\right)_z \Big/ \left(\frac{\partial y}{\partial x}\right)_z. \tag{13}$$

At the Curie point the relaxation times increase without limit.[7]

The results obtained above can be extended to a system with an arbitrary number N of additional internal parameters $\eta_1, \ldots, \eta_N$ by an analogous scheme. In this case we have for the dynamic derivatives

$$\left(\frac{\partial y}{\partial x}\right)_{z,\omega} = \left(\frac{\partial y}{\partial x}\right)_z \prod_{n=1}^{N} \frac{1 + i\omega\tau_n^{yz}}{1 + i\omega\tau_n^{xz}}, \tag{14}$$

where τ_n^{yz} and τ_n^{xz} are the relaxation times of the additional parameters for constant y, z and constant x, y respectively. These times relate to the additional parameters with the same index n, provided that $\eta_1, \ldots, \eta_N$ are independent. In the opposite case the relaxation times appear to be general ones and cannot be associated with individual parameters. In particular, the approach of the η_n to their equilibrium values $\eta_n^{(0)}$ for constant x, y will take place according to the law

$$\eta_n = \eta_n^{(0)} + \sum_{k=1}^{N} A_{nk}^{xy} \exp(-t/\tau_k^{xy}), \tag{15}$$

where the coefficients A depend on the initial conditions.

Equations (10) and (11) can be applied to the description of all relaxation phenomena connected with the corresponding dynamic derivatives. The thermodynamical theory leaves open the question of the temperature dependence of the relaxation times.

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A POSSIBLE TEST OF THE CONSERVATION OF "COMBINED PARITY" IN THE BETA INTERACTION

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It is suggested that the invariance of the β-interaction under time reversal can be examined by observing $\beta-\gamma$ angular correlations in allowed transitions with aligned nuclei. Formulas are obtained for the $\beta-\gamma$ correlation in nuclei oriented by various methods. The general form of the β-interaction is considered with parity nonconservation taken into account. An expression is given for the $\beta-\gamma$ angular and polarization correlations in oriented nuclei for β-transitions of any order of forbiddenness.

EXPERIMENT has at present verified that parity is not conserved in weak interactions, as had been hypothesized in different forms by Lee and Yang[1,2] and by Landau.[3,4] It remains for experiment to test the specific versions of the hypothesis. We shall speak here primarily of a test of the conservation law for "combined parity," which according to Pauli's theorem[5] is equivalent to the invariance under time reversal.

Another important question is whether or not is is possible to describe the neutrino by a two-component equation.[4,2]

In the present work we shall give our main consideration to the first of these problems. We shall do this by investigating the $\beta-\gamma$ angular correlation of oriented nuclei. We shall show that if the nuclei are oriented by the method of Bleaney[6] or Pound[7] (aligned nuclei), then a measurement of the $\beta-\gamma$ correlation for allowed transitions can give important information on the nature of the β-interaction:

(1) If there is no $\beta-\gamma$ angular correlation, then (a) the β-interaction is invariant with respect to

time reversal, and (b) the vector and pseudovector interactions contribute weakly.

(2) If there is a $\beta-\gamma$ angular correlation, but it decreases with an increase in the β energy, then (a) the β-interaction is invariant with respect to time reversal, but (b) either the vector or pseudovector interactions contribute strongly.

(3) An increase in the correlation with an increase in the β energy leads uniquely to the conclusion that the β-interaction is not invariant under time reversal, and "combined parity" is not conserved.

These results are easily understood if one bears in mind the following facts. The expression for the probability of the β-transition must contain the electron momentum $\mathbf{p}(p, \vartheta, \varphi)$, the photon momentum $\mathbf{k}(k, \theta, \phi)$, and the tensor $T_{i\ell}$ which determines the orientation of the aligned nucleus. Since the transition is allowed, $\mathbf{p}$ must enter linearly. Parity is conserved in the γ-transition, so that $\mathbf{k}$ must enter quadratically. With these conditions, it is impossible to construct a scalar quantity of $\mathbf{p}$, $\mathbf{k}$, and $T_{i\ell}$; the only pseudoscalar that can be constructed of these quantities is $[\mathbf{p}\times\mathbf{k}]\mathbf{T}$, where $T_\ell = \sum_i T_{i\ell}k_i$. This pseudoscalar changes sign under time reversal, and therefore cannot enter a theory which is invariant with respect to the transformation $t \to -t$. The Coulomb interaction of the electron and the nucleus leads to additional terms in the expression for the $\beta-\gamma$ angular correlation. In allowed transitions these terms enter only into the vector and axial vector interactions, and they decrease as the electron energy increases.

Let us choose the most general form of the β-interaction, taking parity nonconservation into account by the method of Lee and Yang.[1] The initial state of the nucleus shall be given by the tensor $P^{j_0}_{g\eta}$, which is related to the density matrix $\rho(\mu_0, \mu_0')$ by

$$P^{j_0}_{g\eta} = \frac{2g+1}{2j_0+1}\sum_{\mu_0, \mu_0'} C^{j_0\mu_0'}_{j_0\mu_0 g\eta}\,\rho(\mu_0, \mu_0'), \tag{1}$$

where μ_0 is the projection of the nuclear spin j_0 on the axis of quantization. If this axis is chosen as the physically defined axis of the most probable nuclear spin orientation (for instance directed along the external orienting field), then the density matrix is diagonal, and the nuclear orientation is given by

$$P^{j_0}_{g0} = \frac{2g+1}{2j_0+1}\sum_{\mu_0} C^{j_0\mu_0}_{j_0\mu_0 g0}\,w(\mu_0), \tag{2}$$

where $w(\mu_0)$ is the probability of finding a nucleus in a state with the z component of the spin equal to μ_0. We choose the initial wave function of the system as

$$\Psi_i = \sum_{\mu_0} a^{\xi}_{\mu_0}\psi_{j_0\mu_0}, \quad \rho(\mu_0, \mu_0') \doteq \langle a^{\xi*}_{\mu_0} a^{\xi}_{\mu_0'}\rangle_{\xi},$$

where $\langle\ \rangle_\xi$ denotes averaging over the elements of the statistical ensemble, and

$$w(\mu_0) = \langle |a^{\xi}_{\mu_0}|^2\rangle_\xi, \quad \sum_{\mu_0} w(\mu_0) = 1.$$

If after β-decay the nucleus undergoes a radiative transition, the polarization tensor of the intermediate state can be expressed in terms of observed quantities which characterize this latter transition. Let $H_{j_1\mu_1}$ be the matrix element for the γ-transition from state $j_1\mu_1$ to state $j_2\mu_2$. Then the polarization tensor for state $j_1\mu_1$ can be written

$$P^{j_1}_{S\sigma} = \frac{2S+1}{2j_1+1}\sum_{\mu_1, \mu_1'} C^{j_1\mu_1}_{j_1\mu_1' S\sigma} H_{j_1\mu_1} H^*_{j_1\mu_1'}. \tag{3}$$

We shall choose the form used previously by the author and others[8,9] for the wave function of the electron outside the nucleus, a function which describes a particle with momentum p at infinity. We shall define the wave function of the electron within the nucleus as the solution of the Dirac equation for a surface or volume charge distribution. The amplitude of the internal function will be determined from the condition that the internal and external functions join smoothly. The parameters $A_{j\lambda}$, which are adjusted to make these functions joint smoothly, and the wave function phases $\delta_{j\lambda}$ have been previously determined by the author[9 - 11] and have been tabulated by Sliv and Volchok.[12]* We shall denote the total angular momentum of the electron by j, and its orbital angular momentum by $l_1 = j + \lambda$, where $\lambda = \pm \frac{1}{2}$. The total angular momentum of the neutrino will be denoted by i, and its orbital angular momentum by $l_1 = i + \nu$, where $\nu = \pm \frac{1}{2}$. The matrix elements of the $\beta - \gamma$ transition are calculated in the same way as in the author's previous works.[9,10] As a result we obtain an expression for the angular and polarization correlations of the β particle and the γ ray emitted by the oriented nuclei. This expression is

$$W(\mathrm{p}, \mathrm{k}) = \sum i^{l+l'} (-1)^{j'-j+g} \sqrt{(2l+1)(2l'+1)/(2f+1)}(2j+1)(2j'+1)\, C^{f0}_{l'0l0}\, W(iL'jf;\, j'L)\, W\left(\tfrac{1}{2}\, jl'f;\, lj'\right)$$
$$\times \exp[i(\delta_{j\lambda} - \delta_{j'\lambda'})]\, C^{f\sigma}_{g0S\sigma} X(j_1j_1S,\, j_0j_0g,\, L'L\, f)\, P^{j_0}_{g_0} P^{j_1}_{S\sigma}(\theta\Phi)\, Y^*_{f\sigma}(\vartheta\varphi)\, B^{L\tau}_{\lambda\nu} B^{L'\tau'*}_{\lambda'\nu}. \qquad (4)$$

Here $C^{c\gamma}_{a\alpha b\beta}$ are Clebsch-Gordan coefficients, W (abcd; ef) are Racah functions, and X (abc, def, ghi) = X (adg, beh, cfi) are Fano functions. These functions are tabulated.[13 - 16] Further

$$B^{L\tau}_{\lambda\nu} = (C^{j_0\mu_0}_{L\Lambda j_1\mu_1})^{-1} \sum_{J=0}^{1} \int \psi^*_{j_1\mu_1} S^L_{J\tau} Y^{J\tau*}_{L\Lambda} r^{L+\tau} \psi_{j_0\mu_0} d\mathrm{r},$$
$$S^L_{J\tau} = \{[\beta(g_S R_1^- + iG_S R_2^+) + (g_V R_1^+ + iG_V R_2^-)]\delta_{J0} + [\beta\sigma(g_T R_1^- + iG_T R_2^+) + \sigma(g_A R_1^+ + iG_A R_2^-)]\delta_{J1}\}\delta_{L+\tau,N} \qquad (5)$$
$$- \{[\gamma_5(ig_A R_2^- + G_A R_1^+) - \beta\gamma_5(ig_P R_2^+ + G_P R_1^-)]\delta_{J0} + [\alpha(ig_V R_2^- + G_V R_2^+) - \beta\alpha(ig_T R_2^+ + G_T R_1^-)]\delta_{J1}\}\delta_{L+\tau,N-1}.$$

The symbols g_S, g_T, g_P, g_V, and g_A denote the coupling constants of the scalar, tensor, pseudoscalar, vector, and pseudovector β-interactions, and G_S, G_T, etc., are analogous coupling constants for terms which can be allowed only in the case of parity nonconservation (that is, those containing an extra $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$). Further, β, γ_5, $\gamma_4 = -\beta$, α, and σ are the well-known Dirac matrices which act on the nuclear wave functions $\psi_{j_0\mu_0}$ and $\psi_{j_1\mu_1}$, and $Y^{J\tau}_{L\Lambda}$ is a (J − L)-vector (with $J = \frac{1}{2}$ this is a spherical spinor). The components of this vector are given by

$$[Y^{J\tau}_{L\Lambda}]_\gamma = (-1)^{J-\gamma}\, C^{L\Lambda}_{f\varphi J-\gamma} Y_{f\varphi}, \quad f = L + \tau, \quad -J \leqslant \tau \leqslant J. \qquad (6)$$

The spherical function $Y_{f\varphi}$ are given here in Bethe's definition,[17] which differs by a factor of $(-1)^\varphi$ from the definition of Landau and Lifshitz.[18] The quantities $R_1^\pm$ and $R_2^\pm$ arise from the wave functions of the light particles, and are

$$R_1^\pm = \sum_{\omega,\chi} [\delta_{\omega,-\lambda}\delta_{\chi,-\nu} \pm 4\lambda\nu\delta_{\omega,\lambda}\delta_{\chi,\nu}]\, Q_{\omega\chi}, \quad R_2^\pm = \sum_{\omega,\chi} [2\nu\delta_{\omega,-\lambda}\delta_{\chi,\nu} \pm 2\lambda\delta_{\omega,\lambda}\delta_{\chi,-\nu}]\, Q_{\omega\chi},$$
$$Q_{\omega\chi} = A_{j\lambda}\beta^{jiL}_{\omega\chi\tau} \frac{\sqrt{W + 4\omega\lambda}\, \varkappa^{j+\omega}(-q)^{i+\chi}}{(2j+2\omega+1)!!\,(2i+2\chi+1)!!}\, \varphi_\omega \delta_{j+i+\omega+\chi,\, L+\tau}, \qquad (7)$$

$\omega = \pm \frac{1}{2}$, $\chi = \pm \frac{1}{2}$, q is the neutrino momentum, W = E + V, $\kappa^2 = W^2 - 1$, E is the total energy of the electron including its rest mass,† $V = \alpha Z/R$ for a surface charge distribution, and $V = 3\alpha Z/2R$ for a uniform volume charge distribution. The quantity φ_ω is equal to unity with an accuracy of $(\alpha Z)^2/5$. In the case of a surface distribution, for instance,

$$\varphi_\omega = 1 + 3(\varkappa R)^2[10(2j + 2\omega + 3)]^{-1}.$$

In the case of a volume distribution, φ_ω differs from unity by even less.[9,10] Finally, R is the nuclear radius, and $\alpha = 1/137$.

The $\beta^{jiL}_{\omega\chi\tau}$ are the expansion coefficients of the angular parts of the wave functions of the light particles, i.e.,

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*The quantities $a_{j\lambda}$ which are tabulated by Sliv and Volchok[12] differ from the $A_{j\lambda}$ defined by the author[9 - 11] by the factor

$$a_{j\lambda} = \sqrt{W - 2\lambda}\, \varkappa^j A_{j\lambda}/(2j)!!$$

†We are using units throughout in which ħ = c = m = 1.

$$(Y_{j\mu}^{1/2\lambda *} A_{J\gamma} Y_{in}^{1/2\nu}) = \sum_{L,\tau} \beta_{\lambda\nu\tau}^{jiL} C_{L\Lambda in}^{j\mu} (-1)^{J+\tau+\Lambda} [Y_{L-\Lambda}^{J\tau}]_{\gamma},$$

$$\beta_{\lambda\nu\tau}^{jiL} = \sqrt{(2J+1)(2L+1)(2f+1)(2l_1+1)(2i+1)/2\pi}\; C_{l_1 0 f 0}^{l_0} X(1/2\, lj,\ 1/2\, l_1 i,\ JfL), \tag{8}$$

$A_{00} = 1$, $A_{1\gamma} = \sigma_\gamma$ is a Pauli matrix, $\sigma_0 = \sigma_z$, $\sigma_{\pm 1} = \pm(\sigma_x \pm i\sigma_y)/\sqrt{2}$, $\ell = j + \lambda$, $\ell_1 = i + \nu$, and $f = L + \tau$. The explicit form of the $\beta_{\omega\chi\tau}^{jiL}$ is easy to find from the tables of Ref. 16. The sum in (4) is taken over all possible values of all indices. Equation (4) is applicable to β-transitions of all orders of forbiddenness. The index N in Eq. (5) gives the order of forbiddenness of the transition. For allowed transitions $N = 0$, and since $L + \tau \geq 0$, the second curly bracket gives no contribution to $S_{J\tau}^{L}$. If one treats a transition of a definite order of forbiddenness, only some of the terms remain in the sum in Eq. (4), since the Clebsch-Gordan coefficients, the Racah functions, and the Fano functions in $S_{J\tau}^{L}$ are different from zero only for certain values of the indices. The number of possible values of g is determined by the character of the nuclear orientation.

For aligned nuclei, g is even. If one measures the angular distribution of the γ-rays, then

$$P_{S\sigma}^{j_1}(\theta,\ \Phi) = \left[1 - \frac{S(S+1)}{2I(I+1)}\right]\sqrt{(2S+1)(2I+1)(2j_1+1)}\, W(j_2 I j_1 S;\ j_1 I)\, C_{I0S0}^{I0} Y_{S\sigma}(\theta\Phi), \tag{9}$$

where I is the multipole order of the quantum emitted, S is even, and j_1 and j_2 are the nuclear angular momenta before and after the radiative transition. If the nuclei are oriented by the method of Bleaney[6] or Pound[7] then g is even. For this case (4) gives

$$W_2(\mathrm{p},\mathrm{k}) = \sum_{S=0,2,\ldots} \sum_{g=0,2,\ldots} \sum_{\mu_0} \sqrt{2g+1}\, C_{j_0\mu_0 g0}^{j_0\mu_0} w(\mu_0) \Big\{\sqrt{2S+1} \sum_{L=0}^{1} [M_L - \gamma_1 E^{-1} N_L]\, \delta_{S,g}\, U(Lj_0 j_1 g;\ j_1 j_0)\, P_S(\cos\theta)$$

$$- \sqrt{2g+1}\,[p \operatorname{Im} Q_m - \alpha Z \operatorname{Re} Q_n]\, E^{-1} U(j_0 j_0 g 1;\ S j_0)\, \frac{2}{3}\, iF_{Sg1}(\vartheta\varphi\theta\Phi)\Big\}\left[1 - \frac{S(S+1)}{2I(I+1)}\right] C_{I0S0}^{I0}\, U(j_2 I j_1 S;\ j_1 I), \tag{10}$$

$$M_0 = (|g_S|^2 + |G_S|^2)\,|K_S|^2 + (|g_V|^2 + |G_V|^2)\,|K_V|^2, \quad M_1 = (|g_T|^2 + |G_T|^2)\,|K_T|^2 + (|g_A|^2 + |G_A|^2)\,|K_A|^2,$$

$$N_0 = (g_S g_V^* + G_S G_V^*)\, K_S K_V^* + \text{complex conjugate}, \quad N_1 = (g_T g_A^* + G_T G_A^*)\, K_T K_A^* + \text{complex conjugate},$$

$$Q_m = (g_T G_S^* + G_T g_S^*)\, K_T K_S^* - (g_A G_V^* + G_A g_V^*)\, K_A K_V^*, \quad Q_n = (g_A G_S^* + G_A g_S^*)\, K_A K_S^* - (g_T G_V^* + G_T g_V^*)\, K_T K_V^*, \tag{11}$$

$$F_{abc}(\vartheta\varphi\theta\Phi) = 4\pi \sum_{\gamma} C_{b0a\gamma}^{c\gamma} Y_{a\gamma}(\theta\Phi)\, Y_{c\gamma}^{*}(\vartheta\varphi), \quad F_{0bb} = \sqrt{2b+1}\, P_b(\cos\vartheta), \quad F_{a0a} = 4\pi \sum_{\gamma} Y_{a\gamma}(\theta\Phi)\, Y_{a\gamma}^{*}(\vartheta\varphi), \tag{12}$$

$$iF_{221} = -3\sqrt{3/2}\sin\theta\cos\theta\sin\vartheta\sin(\Phi-\varphi), \quad iF_{441} = -(3\sqrt{5}/4)(7\cos^2\theta - 3)\cos\theta\sin\theta\sin\vartheta\sin(\Phi-\varphi), \tag{13}$$

$$U(abcd;\ ef) = \sqrt{(2e+1)(2f+1)}\, W(abcd;\ ef), \quad \gamma_1 = \sqrt{1-(\alpha Z)^2},$$

$$K_S = \int \psi_{j_1\mu_1}^{*} \beta \psi_{j_0\mu_0}\, dr, \qquad K_T = \sqrt{4\pi}\,[C_{1\Lambda j_1\mu_1}^{j_0\mu_0}]^{-1} \int \psi_{j_1\mu_1}^{*} \beta\sigma Y_{1\Lambda}^{-1} \psi_{j_0\mu_0}\, dr.$$

In these equations K_V and K_A differ from K_S and K_T by the absence of β. For allowed transitions, $K_V = -K_S$, and $K_A = -K_T$. Here $Y_{1\Lambda}^{-1}$ is the vector of Eq. (6) with $J = 1$, $L = 1$, and $\tau = -1$. Equation (10) is accurate up to terms of order $(\alpha Z)^2/3$ and αZER compared with unity.* If high accuracy is necessary, $-\gamma_1 E^{-1}$ should be replaced by a^{+}/a^{-}, where

$$a^{\pm} = (W-1)\, A_{1/2\,1/2}^{2} \pm (W+1)\, A_{1/2\,-1/2}^{2},$$

p should be replaced by c_1/a^{-}, where

$$c_1 = 2\varkappa A_{1/2\,1/2} A_{1/2\,-1/2} \cos(\delta_{1/2\,-1/2} - \delta_{1/2\,1/2})$$

and αZ should be replaced by c_2/a^{-}, where

$$c_2 = 2\varkappa A_{1/2\,1/2} A_{1/2\,-1/2} \sin(\delta_{1/2\,-1/2} - \delta_{1/2\,1/2}).$$

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*Equation (14) will be given to the same accuracy.

Since it seems that invariance under time reversal is maintained in strong nuclear interactions, we may assert that the phase difference between K_T and K_S^*, as well as that between K_A and K_V^*, etc., is either 0 or π. Whether Q_m and Q_n are real or complex is therefore determined by the coupling constants g and G. If the β-interaction is invariant under the transformation $t \to -t$, all the g and G are real and $\operatorname{Im} Q_m = 0$. In this case the $\beta-\gamma$ correlation is independent of p/E. If this independence were observed, it would be direct proof of the breakdown of invariance under time reversal. The form of allowed β-spectra would seem to indicate that the real part of g_A and G_A is either very small or vanishes identically. There is indirect evidence that the real part of g_V and G_V is also small. If this is true, then Q_n is small. Thus even a rough qualitative study of $\beta-\gamma$ angular correlation in aligned nuclei may answer the question of invariance under time reversal and give indications as to the contribution of the V and A interactions.

If the nuclei are polarized, say by the Gorter-Rose method,[19] there will again be a $\beta-\gamma$ correlation for allowed transitions. Considering both even and odd values of g, we obtain

$$W_1(\mathbf{p}, \mathbf{k}) = W_2(\mathbf{p}, \mathbf{k}) - \sum_{S=0,2,\ldots} \sum_{g=1,3,\ldots} \sum_{\mu_0} C^{j_0\mu_0}_{j_0\mu_0 g0}\, \varpi(\mu_0)\,(2g+1)\sqrt{2S+1}\left\{\frac{2}{3}\left[\frac{p}{E}\operatorname{Re} Q_m + \frac{\alpha Z}{E}\operatorname{Im} Q_n\right]\right.$$

$$\times \sqrt{2j_0+1}\,W(j_0 j_0 g 1;\; S j_0) - \sqrt{\frac{2}{3}}\left[\frac{p}{E}\operatorname{Re} Q_1 + \frac{\alpha Z}{E}\operatorname{Im} Q_1\right]\sqrt{(2j_0+1)(2j_1+1)}\,X(j_1 j_1 S,\, j_0 j_0 g,\, 111)\Big\}$$

$$\times F_{Sg1}(\vartheta\varphi\theta\phi)\left[1 - \frac{S(S+1)}{2I(I+1)}\right] C^{I0}_{I0S0}\, U(j_2 I j_1 S;\; j_1 I); \tag{14}$$

$$Q_1 = (G_T g_T^* + G_T^* g_T)\,|K_T|^2 - (G_A g_A^* + G_A^* g_A)\,|K_A|^2 - (G_T g_A^* + g_T G_A^*)\,K_T K_A^* + (G_A g_T^* + g_A G_T^*)\,K_A K_T^*; \tag{15}$$

$$F_{211} = -\sqrt{3/2}\,\{(3\cos^2\theta - 1)\cos\vartheta + 3\sin\theta\cos\theta\sin\vartheta\cos(\phi - \varphi)\},$$

$$F_{231} = (3\sqrt{2}/2\sqrt{7})\,\{(3\cos^2\theta - 1)\cos\vartheta - 2\sin\theta\cos\theta\sin\vartheta\cos(\phi - \varphi)\},$$

$$F_{431} = -(3/4\sqrt{7})\,\{(35\cos^4\theta - 30\cos^3\theta + 3)\cos\vartheta + 5(7\cos^2\theta - 3)\cos\theta\sin\theta\sin\vartheta\cos(\phi - \varphi)\}. \tag{16}$$

A measurement of $W_1(\mathbf{p}, \mathbf{k})$ may serve as a control experiment in the study of $W_2(\mathbf{p}, \mathbf{k})$, and is also interesting in itself for the determination of the g and G coupling constants. The use of the two-component equation[2,4] to describe the neutrino necessitates an experimental proof of the fact that all the coupling constants are related by $g = -G$ or $g = G$. This requires quantitative measurements, so that it is desirable to use the results of independent experiments. In particular, the polarization of electrons emitted by polarized nuclei is of interest. The author has previously[20] derived the required formulas for allowed and forbidden β-transitions. We note that if the nucleus emits β^+ particles, their polarization can be observed not only from scattering asymmetry, but also from the circular polarization of the annihilation radiation. Since the positrons are depolarized by relativistic effects, they will not be completely depolarized in spite of the fact that they are slowed down.

Additional possibilities in the investigation of β processes arise if, in addition to the angular distribution, one observes the angular polarization of the γ-rays emitted after β-decay. It was shown by Shapiro,[21] for instance, that if parity is not conserved there should be a correlation between the direction of the emitted electron and the circular polarization of the γ-ray. The present author[22] has suggested that observations of the circular polarization of x-rays emitted after K-capture be used to determine the sign of g_S/g_T. The circular polarization of the γ-rays gives information on the polarization of the nuclear dipole after β-decay. Thus instead of studying β-transitions in polarized nuclei, one can find the direction of the nuclear spin relative to the direction of the β particle by observing the circular polarization of the subsequent photons. Formally this reduced to the $P^{j_1}_{S\sigma}$ tensor playing the role of $P^{j_0}_{g\eta}$. If S is odd, $P^{j_1}_{S\sigma}$ vanishes on averaging over γ-ray polarizations. If, however, the circular polarization is found from experiment $P^{j_1}_{S\sigma}$ is nonzero for all S, and Eq. (9) becomes

$$P^{j_1}_{S\sigma} = (2S+1)\sum_{M,\,M'} \xi^{(M-M')/2}\, C^{IM}_{IM'SM-M'}\, U(I j_2 S j_1;\; j_1 I)\, e^{i(M-M')\alpha}\, D^S_{\sigma,M-M'}(\phi, \theta, 0), \tag{17}$$

where the term with $M = M' = 1$ gives $P^{j_1}_{S\sigma}$ for right-circularly polarized photons, the term with $M = M'$

$= -1$ gives it for left-circularly polarized photons, and the sum of terms with $M = 1$, $M' = -1$ and $M = -1$, $M' = 1$ gives this tensor for linearly polarized photons. The variable ξ assumes the values $+1$ for magnetic and -1 for electric transitions. The angle α gives the linear polarization vector ϵ in the coordinate system whose z axis is along **k** ($\epsilon_{\pm 1} = \pm e^{\pm i\alpha}/\sqrt{2}$, $\epsilon_0 = 0$). This angle α is the angle between the polarization vector and the x axis in the plane perpendicular to **k**. Finally, $D^S_{\sigma, M-M'}(\phi\theta 0)$ is a matrix of the irreducible representation of the rotation group.[23] It is here defined so that $\psi_{j\mu}(\theta\phi) = \sum_{\mu'} D^j_{\mu\mu'}(\phi\theta 0)\psi_{j\mu'}(00)$, and in particular,

$$Y_{lm}(\theta\phi) = \sqrt{(2l+1)/4\pi}\,D^l_{m0}(\phi\theta 0).$$

Experiments on allowed β-transitions give the most unambiguous information, and are therefore of most interest. In the case of forbidden β-transitions, Eq. (4) depends on more nuclear matrix elements than it does in the case of allowed transitions. These matrix elements must be considered as additional unknown quantities which must be determined by experiment or evaluated roughly with the aid of nuclear models. It is therefore difficult to give unique interpretations to the experimental data. More meaningful conclusions can be reached in studying transitions in which the nuclear angular momentum change is one unit greater than the order of forbiddenness. In this case, however, we obtain no data on the Fermi part of the β-interaction. We shall give here an explicit expression for the $\beta-\gamma$ correlation function for a polarized nucleus undergoing first forbidden transitions, when $j_1 = j_0 \pm 2$. According to Eq. (4) we have

$$W(\mathrm{p}, \mathrm{k}) = \sum_{j,f,g,s}\ \sum_{l' \geqslant l}\ \sum_{\mu_0} C^{j_0\mu_0}_{j_0\mu_0 g0}\, w(\mu_0)(-1)^g X(j_1 j_1 S, j_0 j_0 g,, 22f)(2g+1)$$

$$\times \gamma^{jf}_{ll'} Z^j_{ll'} F_{Sgf}(\vartheta\varphi\theta\Phi)\sqrt{2S+1}\left[1 - \frac{S(S+1)}{2I(I+1)}\right] C^{I0}_{I0S0}\, U(j_2 I j_1 S;\ j_1 I), \tag{18}$$

$$Z^{1/2}_{01} = 2q^2\,[(p\,\mathrm{Re}\,Q_1 + \alpha Z\,\mathrm{Im}\,Q_1)\,\omega_+ - i\,(p\,\mathrm{Im}\,Q_1 - \alpha Z\,\mathrm{Re}\,Q_1)\,\omega_-],$$

$$Z^{3/2}_{12} = [p^2 + (\alpha Z E)^2]\left[\left(p\,\mathrm{Re}\,Q_1 + \frac{\alpha Z}{2}\mathrm{Im}\,Q_1\right)\omega_+ - i\left(p\,\mathrm{Im}\,Q_1 - \frac{\alpha Z}{2}\mathrm{Re}\,Q_1\right)\omega_-\right];\quad Z^{1/2}_{00} = q^2(E+1)(M_1 - N_1),$$

$$Z^{1/2}_{11} = q^2(E-1)(M_1 + N_1),\quad Z^{3/2}_{11} = \frac{p^2 + (\alpha Z E)^2}{2}(E+1)(M_1 - N_1),\quad Z^{3/2}_{22} = \frac{p^2 + (\alpha Z E)^2}{2}(E-1)(M_1 + N_1), \tag{19}$$

$$j = {}^1/_2,\ {}^3/_2;\quad l \text{ and } l' = j \pm {}^1/_2;\quad 2\omega_\pm = 1 \pm (-1)^{f+S+g};$$

$$\gamma^{jf}_{ll'} = (-1)^l (2j+1)\sqrt{(2l'+1)/4(2f+1)}\,C^{f0}_{l'0l0}\, U(2-j, 2, j, f;\ j, 2)\, U({}^1/_2\ jl'f;\ lj), \tag{20}$$

$$\gamma^{1/2\,0}_{ll} = -\sqrt{6}\,\gamma^{1/2\,1}_{01} = \frac{1}{2}\,\gamma^{3/2\,0}_{ll} = 1,\quad \gamma^{3/2\,2}_{11} = \gamma^{3/2\,2}_{22} = \sqrt{7/3}\,\gamma^{3/2\,1}_{12} = -(7/6)\,\gamma^{3/2\,3}_{12} = -\sqrt{14}/5.$$

The quantities M_1, N_1, and Q_1 differ from those in (11) and (15) only in that K_S and K_V are zero, and K_T and K_A are given by

$$K_T = -K_A = [C^{j_0\mu_0}_{2\Lambda j_1\mu_1}]^{-1}\int \psi^*_{j_1\mu_1}\beta\sigma \mathbf{Y}^{-1*}_{2\Lambda}\psi_{j_0\mu_0}\,d\mathrm{r}. \tag{21}$$

All the equations in this article refer to β^--decay. For positron decay, the following substitutions must be made:

$$g_S \rightarrow -g^*_S,\ g_A \rightarrow -g^*_A,\ g_P \rightarrow -g^*_P,\ g_T \rightarrow g^*_T,\ g_V \rightarrow g^*_V,$$
$$G_S \rightarrow G^*_S,\ G_A \rightarrow G^*_A,\ G_P \rightarrow G^*_P\ G_T \rightarrow -G^*_T,\ G_V \rightarrow -G^*_V,\ \alpha Z \rightarrow -\alpha Z.$$

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CONTRIBUTION TO THE DECAY THEORY OF A QUASI-STATIONARY STATE

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Some general problems relating to the decay theory of a quasi-stationary state are considered. Dispersion relations are derived on the basis of the semi-finiteness of the energy distribution density $\omega(E)$. A criterion of physical feasibility in decay theory is formulated and studied on the basis of the Paily and Wiener theorem. It is shown that an exponential decay law cannot hold for all $\Gamma t/\hbar$. Corrections to the exponential decay law are computed under the simplest assumptions. Dispersion relations between the modulus and the phase of the function $p(t)$ are derived and investigated. On the basis of a knowledge of the decay law, these allow us to determine the energy distribution density $\omega(E)$ analytically. The results obtained are derived from the general laws of quantum mechanics and do not depend on the model of the decaying system.

IN the present work we consider several problems of decay theory, in particular, the decay of a quasi-stationary (almost stationary) state.* As is well known, the theory of the decay of a quasi-stationary state has great significance in the investigation of α-decay, in the transmission of particles through a potential barrier, in the theory of the nucleus, in the determination of the distribution of energy levels, etc.[1,2] The basic theorem of decay theory of a quasi-stationary state was obtained by Fock and Krylov in the study of the uncertainty relation for energy and time.[1] A theoretical proof of the observed exponential decay law was obtained in that paper, and the well-known dispersion formulas for the energy distribution of the decaying system were derived.

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*A brief account of results that have been obtained is given in Ref. 5.

The present work is a further study of the decay theory of the quasi-stationary state. The study is based on the theory of the Fourier transform in the complex plane,[3] which has already been used in the study of dispersion relations.[4]

The fundamental assumptions of decay theory of the quasi-stationary state, and the results obtained in the work of Krylov and Fock,[1] are treated briefly in the first section. In the second section, the general dispersion relations in decay theory are obtained. In the third section, a criterion of physical feasibility in decay theory is formulated and investigated. Several fundamental questions are also considered in connection with this fact. Finally, the definitive "dispersion" relations and their detailed investigation are treated in the fourth section.

1. BASIC PREMISES OF THE DECAY THEORY OF A QUASI-STATIONARY STATE

Let $\psi_0 = \psi(x, 0)$ be a wave function describing the state of a physical system at the time $t = 0$. Then the state of this system at the time t is determined by the wave function $\psi(x, t)$*

$$\psi(x, t) = \int e^{-iEt/\hbar} c(E)\, \psi_E(x)\, dE, \tag{1.1}$$

with

$$\psi_0 = \psi(x, 0) = \int c(E)\, \psi_E(x)\, dE, \tag{1.2}$$

while $\psi_E(x)$ are the eigenfunctions of the energy operator of the physical system.

Investigation of the behavior of the function $\psi(x, t)$ in its dependence on the variable t, which is the most general problem pertaining to the time development of the system, can be carried out by starting from certain general principles and restrictions applied to the coefficients $c(E)$ and the eigenfunctions $\psi_E(x)$.

One of the basic problems which arises in the study of the development of a system according to the Schrödinger equation consists in the consideration of the decay of the initial state. But this problem amounts precisely to the determination of the probability that the system at the time t will still be in the initial state $\psi_0 = \psi(x, 0)$. The desired probability is then the projection of the state vector $\psi(x, t)$ on the initial state vector $\psi_0(x, 0)$, and is consequently equal to the square of the modulus of the scalar product

$$p(t) = (\psi(x, t), \psi(x, 0)) = \int \psi(x, 0)\, \psi(x, t)\, dx, \tag{1.3}$$

so that the probability that the physical system at the time t will still be in the initial state ψ_0 is

$$L(t) = |p(t)|^2, \tag{1.4}$$

where in turn, on the basis of (1.1) and (1.2),

$$p(t) = \int e^{-iEt/\hbar} c^*(E)\, c(E)\, dE = \int e^{-iEt/\hbar} \omega(E)\, dE, \tag{1.5}$$

while $\omega(E)$ is the energy distribution density for the initial state ψ_0 (and consequently also for the state at the time t).

The basic theorem of the decay theory (pointed out above), giving the connection between the decay law of the quasi-stationary state and the energy distribution density in this state, was obtained in Ref. 1.

Inasmuch as the condition for the decay of the initial state ψ_0 is evidently the condition

$$L(t) \to 0 \text{ for } t \to \infty, \tag{1.6}$$

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*We shall keep the same notation as used in the paper of Kyrlov and Fock.[1] In particular, the set of variables in terms of which (exluding the time t) the wave function of the system is expressed, is denoted by the letter x.

then it follows from (1.5) (because of the Fourier theorem and the theorem on characteristic functions) that the necessary and sufficient condition for the decay is the absolute integrability of the energy distribution density $\omega(E)$.[1] In particular, it follows from this that if we have a discrete spectrum, then the decay is possible only if all the probabilities pertaining to a discrete spectrum vanish in the initial state; consequently, the $\omega(E_i)$ also vanish. This principal limitation is of advantage to us in what follows.

We note that the determination of the decay probability as the projection of the state vector $\psi(x, t)$ on the initial state vector ψ_0 is most natural and general. It is clear that in the determination of the decay it is necessary to point out the state the decay of which is studied, which leads to a natural determination of the initial instant of time $t = 0$.

We can arbitrarily define the decay probability as the square of the modulus of the wave function $\psi(x, t)$ for a certain fixed value of x. As will be shown, the results obtained in the present research are also applicable for such an "arbitrary" determination of the decay probability.

It was shown in Ref. 1 that relative to the energy distribution density $\omega(E)$ of the decaying state, we can frequently exhibit more than merely the property of absolute integrability. In particular, in the problem of the emission of a particle from a potential well through a potential energy barrier (α-decay) the energy distribution density $\omega(E)$ is a meromorphic function of the complex variable E.

Then, taking into account only the pair of poles close to the real axis,

$$E = E_0 \pm i\Gamma;\ E_0 > 0;\ \Gamma > 0, \tag{1.7}$$

we can, for sufficiently large times t, obtain the following formula:[1]

$$p(t) = \exp\left\{-\frac{iE_0 t}{\hbar} - \frac{\Gamma}{\hbar}|t|\right\}, \tag{1.8}$$

and then

$$L(t) = e^{-2\Gamma|t|/\hbar}, \tag{1.9}$$

which expresses the usual, well-known exponential law of decay of the quasi-stationary state. In this case, $\omega(E)$ is evidently determined by the expression

$$\omega(E) = \frac{1}{\pi}\Gamma/((E - E_0)^2 + \Gamma^2), \tag{1.10}$$

i.e., a widely-used dispersion formula for the energy distribution of the quasi-stationary state is obtained for $\omega(E)$. In what follows, we shall consider some fine points in the derivation of Eq. (1.8).

Concluding this section, we note that the fundamental theory of Fock and Krylov [Eqs. (1.4) and (1.5)] has very general meaning and is based only on the fundamental definitions and concepts of the quantum theory; at the same time, it is completely independent of the concrete form of the Hamiltonian operator. i.e., of the concrete properties of the decaying system.

2. GENERAL "DISPERSION" RELATIONS IN DECAY THEORY

For the following investigation it is necessary to make several basic formulas of the previous section more specific. That is, we must consider (taking into account the note in the previous section) that all the integrals in E are taken in the region of a continuous positive energy spectrum $E \in (0, \infty)$. Then the basic formula (1.5) takes the form

$$p(t) = \int_{-\infty}^{\infty} e^{-iEt/\hbar}\,\bar{\omega}(E)\,dE, \tag{2.1}$$

where the function $\bar{\omega}(E)$ that we have introduced is "semi-finite:"

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E > 0 \\ 0 & E \leqslant 0. \end{cases} \tag{2.2}$$

From (2.1), we get

$$\bar{\omega}(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{iEt/\hbar}\,p(t)\,dt, \tag{2.3}$$

so that the knowledge of the complex function $p(t)$ allows us to find analytically the energy distribution density $\overline{\omega}(E)$ in accordance with (2.3), and therefore to find the spectrum of the Hamiltonian operator of the decaying system.

Restriction of (2.2) to the admissible distribution density $\overline{\omega}(E)$ adds definite restrictions to the function $p(t)$ and consequently to the function $L(t)$, which has the direct physical meaning of a decay probability.

Actually, transforming to the complex $t = t_1 + it_2$ in

$$p_1(t) \equiv p(-t) = \int_{-\infty}^{\infty} e^{iEt/\hbar}\,\overline{\omega}(E)\,dE \tag{2.4}$$

we get

$$p_1(t) = \int_{-\infty}^{\infty} e^{iEt_1/\hbar}\,e^{-Et_2/\hbar}\,\overline{\omega}(E)\,dE, \tag{2.5}$$

and it is easy to see that (2.5) defines $p_1(t)$ as an analytical function in the upper half plane $\operatorname{Im} t = t_2 > 0$ (Ref. 6), which is the direct consequence of the "semi-finiteness" of $\overline{\omega}(E)$ [Eq. (2.2)]. The analytic character of $p_1(t)$ in the upper half plane imposes substantial limitations on the form of the function $p_1(t)$ in the case of real t, since, on the basis of the Cauchy formula, we evidently obtain*

$$\frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{p_1(t')}{t'-t}\,dt' = p_1(t) + \sum_i \frac{1}{t_i - t}\,\mathrm{Res}\,p_1(t_i), \tag{2.6}$$

where $p_1(t)$ is an analytic function in the upper half plane $\operatorname{Im} t > 0$, t_i are simple poles of the function $p_1(t)$ on the real axis, $\mathrm{Res}\,p_1(t_i)$ are the residues at these poles; the symbol P denotes the integral in the sense of the principal value.

Separating the real and imaginary parts in (2.6), and assuming that $p_1(t)$ does not have any poles on the real axis (a natural physical assumption), we obtain the so-called dispersion relations:

$$\operatorname{Re} p_1(t) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\operatorname{Im} p_1(t')}{t'-t}\,dt', \quad \operatorname{Im} p_1(t) = -\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\operatorname{Re} p_1(t')}{t'-t}\,dt'. \tag{2.7}$$

We shall now transform these dispersion relations. By virtue of the real nature of $\overline{\omega}(E)$ we obtain, obviously,

$$p_1^*(t) = p_1(-t); \quad p(t) = p^*(-t). \tag{2.8}$$

From (2.7) and (2.8) we can, after a number of transformations, obtain the following dispersion relations:

$$\operatorname{Re} p(t) = -\frac{2}{\pi}\,\mathrm{P}\int_{0}^{\infty} \frac{t' \operatorname{Im} p(t') - t \operatorname{Im} p(t)}{t'^2 - t^2}\,dt', \quad \operatorname{Im} p(t) = \frac{2t}{\pi}\,\mathrm{P}\int_{0}^{\infty} \frac{\operatorname{Re} p(t') - \operatorname{Re} p(t)}{t'^2 - t^2}\,dt'. \tag{2.9}$$

Thus, the fact that the energy distribution density $\overline{\omega}(E)$ is "semi-finite" leads to the result that the real and imaginary functions of $p(t)$ are interconnected by integral dispersion relations, so that it is impossible to assign $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$ separately and arbitrarily. The dispersion relations make it possible to calculate $\operatorname{Im} p(t)$ from a knowledge of $\operatorname{Re} p(t)$ alone and vice versa. Inasmuch as, on the basis of (2.3), knowledge of $p(t)$ completely determines $\overline{\omega}(E)$, we can select the value of the dispersion relations (2.9) by the following statement: all the information concerning the energy distribution density $\overline{\omega}(E)$ is contained either in the real or in the imaginary part of the function $p(t)$.

From the mathematical point of view, the dispersion relations (2.9) form a system of explicitly solved integral equations in $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$.

The dispersion relations introduced above, as was shown, are the consequence of the "semi-finiteness" in the sense of (2.2), i.e., when the limit of "semi-finiteness" below is zero. In the problem of the emission of a particle from a potential well through a potential barrier, it is obvious that the limit of the "semi-finiteness" is zero.

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*In the derivation it is essentially assumed that $p_1(t)$ falls off sufficiently rapidly as $|t| \to \infty$.

Now let us consider the consequence of the general condition of semi-finiteness:

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E > E_1 \\ 0 & E \leqslant E_1. \end{cases} \tag{2.10}$$

It is easy to see that in this more general case we can introduce the corresponding "dispersion" relations.

Actually, carrying out a change of variable, we obtain

$$p_1(t) = e^{iE_1t/\hbar} \int\limits_{-\infty}^{\infty} e^{iE't/\hbar} \bar{\omega}_1(E')\,dE' \equiv e^{iE_1t/\hbar} \varphi_1(t;\,E_1), \tag{2.11}$$

where $\bar{\omega}_1(E')$ is already "semi-finite" in the sense of (2.2). But then, according to what was pointed out above, the "dispersion" relations (2.7) are valid for the analytic function $\varphi_1(t;\,E_1)$.

On the basis of (2.7), (2.9), and (2.11), we then get the following "dispersion" relations:

$$\operatorname{Re} p(t) \cos\frac{E_1}{\hbar} t - \operatorname{Im} p(t) \sin\frac{E_1}{\hbar} t$$

$$= -\frac{2}{\pi} \mathrm{P} \int\limits_0^{\infty} \left\{ t' \operatorname{Re} p(t') \sin\frac{E_1}{\hbar} t' + t' \operatorname{Im} p(t') \cos\frac{E_1}{\hbar} t' - t \operatorname{Re} p(t) \sin\frac{E_1}{\hbar} t - t \operatorname{Im} p(t) \cos\frac{E_1}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2},$$

$$\operatorname{Re} p(t) \sin\frac{E_1}{\hbar} t + \operatorname{Im} p(t) \cos\frac{E_1}{\hbar} t$$

$$= \frac{2t}{\pi} \mathrm{P} \int\limits_0^{\infty} \left\{ \operatorname{Re} p(t') \cos\frac{E_1}{\hbar} t' - \operatorname{Im} p(t') \sin\frac{E_1}{\hbar} t' - \operatorname{Re} p(t) \cos\frac{E_1}{\hbar} t + \operatorname{Im} p(t) \sin\frac{E_1}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}, \tag{2.12}$$

which represent explicitly unsolved integral equations relative to $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$. It is obvious that in the particular case in which $E_1 = 0$, the relations (2.12) transform to (2.9).

It is also easy to see that a condition of semi-finiteness of the type

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E < E_2 \\ 0 & E \geqslant E_2 \end{cases} \tag{2.13}$$

leads to the following "dispersion" relations:

$$\operatorname{Re} p(t) \cos\frac{E_2}{\hbar} t - \operatorname{Im} p(t) \sin\frac{E_2}{\hbar} t$$

$$= \frac{2}{\pi} \mathrm{P} \int\limits_0^{\infty} \left\{ t' \operatorname{Im} p(t') \cos\frac{E_2}{\hbar} t' + t' \operatorname{Re} p(t') \sin\frac{E_2}{\hbar} t' - t \operatorname{Im} p(t) \cos\frac{E_2}{\hbar} t - t \operatorname{Re} p(t) \sin\frac{E_2}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2},$$

$$\operatorname{Im} p(t) \cos\frac{E_2}{\hbar} t + \operatorname{Re} p(t) \sin\frac{E_2}{\hbar} t$$

$$= -\frac{2t}{\pi} \mathrm{P} \int\limits_0^{\infty} \left\{ \operatorname{Re} p(t') \cos\frac{E_2}{\hbar} t' - \operatorname{Im} p(t') \sin\frac{E_2}{\hbar} t' - \operatorname{Re} p(t) \cos\frac{E_2}{\hbar} t + \operatorname{Im} p(t) \sin\frac{E_2}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}. \tag{2.14}$$

It is now easy to introduce the dispersion relations for the modulus (which has direct physical meaning) and phase of the function $p(t)$. Actually, taking it into account that

$$p(t) = M(t) \exp[iN(t)], \qquad \operatorname{Re} p(t) = M(t) \cos N(t); \ \operatorname{Im} p(t) = M(t) \sin N(t), \tag{2.15}$$

it is not difficult to obtain "dispersion" relations on the basis of (2.9), (2.12), and (2.14) that connect $M(t)$ and $N(t)$. For example, from (2.9) we obtain the relation

$$M(t)\cos N(t) = -\frac{2}{\pi} \mathrm{P} \int_0^\infty \frac{t' M(t') \sin N(t') - tM(t) \sin N(t)}{t'^2 - t^2} dt',$$
$$M(t)\sin N(t) = \frac{2t}{\pi} \mathrm{P} \int_0^\infty \frac{M(t') \cos N(t') - M(t) \cos N(t)}{t'^2 - t^2} dt', \quad (2.16)$$

which are explicitly unsolved integral equations relative to M(t) and N(t).

Equations (2.16) permit us, generally speaking, to determine N(t) as a solution of an integral equation, knowing M(t′) for all t′ (i.e., knowing the behavior of the decay probability), and in the same way to determine $\overline{\omega}(E)$ analytically on the basis of (2.3). It is true that the solution of the integral equations for N(t) or M(t) is not generally unique. A detailed study and criticism of the "dispersion" relations is given in Sec. 4 of the present work.

To conclude this section, we note that the "dispersion" relations obtained above are valid even for the wave function

$$\psi(x, t) \equiv \Psi(x, t)\, e^{i\varphi(x,t)}, \quad (2.17)$$

so that the "dispersion" relations for ψ(x, t) are obtained by an obvious substitution in (2.9), (2.12), (2.14), (2.16):

$$\operatorname{Re} p(t) \to \operatorname{Re} \psi(x, t), \qquad \operatorname{Im} p(t) \to \operatorname{Im} \psi(x, t),$$
$$M(t) \to \Psi(x, t), \qquad N(t) \to \varphi(x, t). \quad (2.18)$$

3. CRITERION FOR PHYSICAL FEASIBILITY IN DECAY THEORY

In the previous section, dispersion relations were derived on the basis of the "semi-finiteness" of the energy distribution density $\overline{\omega}(E)$. These established the integral connection between the real and imaginary parts of the function p(t), and therefore between the modulus M(t) (which has direct physical meaning) and the phase N(t). However, as will be shown, the dispersion relations do not exhaust all the consequences of the condition of "semi-finiteness" of the energy distribution density $\overline{\omega}(E)$.

One remaining direct consequence of the condition of semi-finiteness of $\overline{\omega}(E)$ is the criterion of physical feasibility in decay theory, which is analogous to the criterion of physical feasibility already used.[4]*

As was shown in the previous section, the modulus M(t) and the phase N(t) of the function p(t) are connected by the "dispersion" relations, so that it is impossible to give separately and arbitrarily the modulus M(t) and phase N(t). However, since the "dispersion" relations form a single relation, then for its proof it is necessary to know the value of M(t) and N(t), i.e., the entire function p(t). At the same time it is observed that on the basis of the condition of "semi-finiteness" of $\overline{\omega}(E)$, we can point out a criterion by which we need satisfy only the modulus M(t) of the function p(t), which has direct physical meaning because $L(t) = M^2(t)$ is the probability that at the moment t the system is still in the initial state ψ_0, i.e., it is the decay probability.

The statement of the problem is this: let an arbitrary non-negative function M(t) be given; can we find such a real function N(t) that the function $\overline{\omega}(E)$, the Fourier transform of which is

$$p(t) = M(t)\, e^{iN(t)}, \quad (3.1)$$

would be "semi-finite" in the sense

$$\overline{\omega}(E) = \begin{cases} \omega(E) & E > E_0 \\ 0 & E \leqslant E_0. \end{cases} \quad (3.2)$$

It is shown that arbitrary M(t) are unattainable, i.e., for certain M(t) it is impossible to select any real function N(t) so as to satisfy the "semi-finiteness" condition (3.2). The limitations on the admissible function M(t) follow from the fundamental theorem of Paily-Wiener.[3]

In decay theory, this theorem can be formulated in the following fashion. In order that L(t) be the probability of decay of a quasi-stationary state, we must satisfy the criterion

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*The criterion of physical feasibility obtained in the present section is also applicable to the study of ψ(x, t), so that it is sufficient to substitute in the final results Ψ(x, t) for M(t) and φ(x, t) for N(t).

$$\int_{-\infty}^{\infty} \frac{|\log M(t)|}{1+t^2} dt < \infty, \tag{3.3}$$

where $L(t) = M^2(t)$.

The criterion (3.3), which the modulus $M(t)$ of the function $p(t)$ must necessarily satisfy, we call the criterion of physical feasibility[5] in the decay theory of a quasi-stationary state.

We note that while the "dispersion" relations (see Sec. 2) depend essentially on the limits of finiteness, the criterion of physical feasibility (3.3) is generally independent of the limits of finiteness (except that the limits of finiteness E_0 be finite).

Failure to satisfy the condition (3.3), i.e., the criterion of physical feasibility, can be due to the integral (3.3) being improper, either on the upper (lower) limit or owing to singularities of the integrand. In either case, $M(t)$, and consequently $L(t)$, cannot vanish over a certain interval of the variable t. Therefore, it does not hold that

$$L(t) = 0 \text{ for } t \geqslant T_0. \tag{3.4}$$

Inasmuch as the integral (3.3) is absolutely convergent, if it converges at all, then the conditions for the convergence of (3.3) are the Cauchy conditions.[7]

The criterion of physical feasibility (3.3) permits us to make the following remarks relative to the function $M(t)$, and consequently relative to $L(t)$ also.

(a) As $|t| \to \infty$, the convergence of the integral (3.3) is guaranteed by satisfaction of the inequality

$$|\log M(t)| / (1+t^2) < At^{-l}, \tag{3.5}$$

where $A > 0$; $\ell > 1$.

(b) The convergence at certain points t_0: $M(t_0) = 0$; $M(t_0) = \infty$ is guaranteed by fulfillment of the following inequality:

$$|\log M(t)| / (1+t^2) < A_1 (t-t_0)^{-l_1}, \tag{3.6}$$

where $A_1 > 0$; $\ell_1 < 1$.

We note that if $M(t)$ satisfies the conditions of the Paily-Wiener theorem, i.e., the criterion (3.3), then the function obtained by polynomial change of $M(t)$ satisfies this same criterion.

The inequalities (3.5) and (3.6) are of fundamental value in the decay theory of the quasi-stationary state, especially the inequality (3.5).

Actually, the inequality (3.5) means that for $|t| \to \infty$, the inequality

$$L(t) \geqslant A \exp(-\gamma |t|^q), \tag{3.7}$$

($A > 0$; $\gamma > 0$, and $q < 1$), is certainly satisfied in all cases, i.e., there can be in principle no exponential decay law for all $t \in (0, \infty)$. It should be noted that the principal conclusion obtained above is the direct consequence of the "semi-finiteness" of the energy distribution density $\overline{\omega}(E)$ and is completely independent of the specific nature of the decaying system.

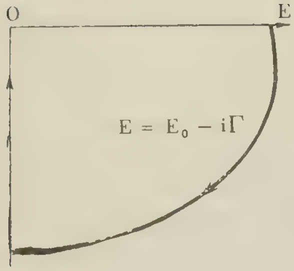
The conclusion thus obtained paradoxically contradicts the well-known experimental exponential decay law. However, as will be shown, in spite of the fact that there can be, in principle, no exponential decay law for all $t \in (0, \infty)$, the exponential decay law is satisfied with sufficient accuracy in a rather large middle region of the values of t.

With this aim, let us consider in detail the derivation of Eq. (1.8), obtained in Ref. 1. In the case in which $\overline{\omega}(E)$ has only poles for $E = E_0 \pm i\Gamma$, $p(t)$ is defined by the integral

$$p(t) = \frac{1}{\pi} \int_0^{\infty} e^{-iEt/\hbar} \frac{\Gamma}{(E-E_0)^2 + \Gamma^2} dE. \tag{3.8}$$

If we consider that $\Gamma \ll E_0$ and $E_0 \gg 0$, then it seems natural to substitute the lower limit in (3.8) by $-\infty$, i.e., approximately

$$p(t) \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iEt/\hbar} \frac{\Gamma}{(E-E_0)^2 + \Gamma^2} dE. \tag{3.9}$$



The integral of (3.9) is easily computed by residues and gives, as in Ref. 1,

$$p(t) \approx e^{-iE_0 t/\hbar} e^{-\Gamma|t|/\hbar}, \tag{3.10}$$

i.e., the exponential decay law

$$L(t) \approx e^{-2\Gamma|t|/\hbar}. \tag{3.11}$$

However, the result (3.10) contradicts the criterion (3.3) which is evidently produced by the approximation made in the computation of (3.8). We consider the more accurate calculation of this integral.

Traversing the closed contour in the lower half plane $\operatorname{Im} E \leq 0$, shown in the picture, we obviously obtain, from the method of residues:

$$p(t) = \exp\{-iE_0 t/\hbar - \Gamma t/\hbar\} - \frac{i}{\pi}\int_0^\infty e^{-Et/\hbar} \frac{\Gamma}{(E_0 + iE)^2 + \Gamma^2}\, dE. \tag{3.12}$$

The first term in (3.9) gives an approximate solution of (3.10). We compute the second term in (3.9). We have

$$A \equiv \frac{i}{\pi}\int_0^\infty e^{-Et/\hbar} \frac{\Gamma}{(E_0 + iE)^2 + \Gamma^2}\, dE = \frac{1}{2\pi}\left\{\int_0^\infty e^{-Et/\hbar} \frac{dE}{E_0 + iE - i\Gamma} - \int_0^\infty e^{-Et/\hbar} \frac{dE}{E_0 + iE + i\Gamma}\right\}. \tag{3.13}$$

In the first integral we make the substitution of variables $E_0 + iE - i\Gamma \equiv z$, and in the second integral, $E_0 + iE + i\Gamma \equiv z$; we obtain from (3.13)

$$A = \frac{i}{2\pi}[f(y_1) - f(y_2)], \tag{3.14}$$

where

$$f(y) = e^{-iy}\int_y^{i\infty} e^{ix}\frac{dx}{x}, \tag{3.15}$$

$$y_1 = (E_0 - i\Gamma)t/\hbar, \qquad y_2 = (E_0 + i\Gamma)t/\hbar. \tag{3.16}$$

Integrating by parts in (3.15) and assuming that

$$(t/\hbar)\sqrt{E_0^2 + \Gamma^2} \gg 1, \tag{3.17}$$

we get

$$f(y) \approx i/y, \tag{3.18}$$

and then, substituting in (3.14), we obtain

$$A = \frac{i}{\pi}\frac{\Gamma\hbar}{(E_0^2 + \Gamma^2)t}. \tag{3.19}$$

Thus, we finally get for values of t that satisfy (3.17)

$$p(t) = \exp\{-iE_0 t/\hbar - \Gamma t/\hbar\} - \frac{i}{\pi}\frac{\Gamma\hbar}{(E_0^2 + \Gamma^2)t}. \tag{3.20}$$

The presence of the second, non-exponential term in (3.20) once again supports the criterion of physical feasibility (3.3), (3.7), since it follows in every case from (3.20) that for $t \to \infty$, $|p(t)|$ vanishes more slowly than exponentially. It also follows from (3.20) that the exponential law of decay is valid only in the finite region of values of $\Gamma t/h$.

$$(\Gamma t/\hbar)\, e^{-\Gamma t/\hbar} \gg (\Gamma/E_0)^2. \tag{3.21}$$

We can show that similar results are obtained if $\overline{\omega}(E)$ differs from (1.10) by the factor $\sqrt{E}$. For brevity, we omit the calculations and write down the final result. The additional non-exponential term in this case is proportional to $t^{-3/2}$ while the condition (3.21) is replaced by

$$(\Gamma t/\hbar)^{3/2} e^{-\Gamma t/\hbar} \gg (\Gamma/E_0)^{3/2}. \tag{3.22}$$

It is evident from the estimates obtained that the difference between the decay laws and the exponential can be observed only for very large or rather small values of t. Here the presence of non-exponential terms naturally leads to the necessity of fixing the initial instant of time $t = 0$. In the following section, we shall touch on the experiments possible in this sense.

The criterion of physical feasibility that has been obtained, in addition to its principal value, is important for the investigation of "dispersion" relations in which the integrals generally extend over infinite limits

To conclude this section, we give (without proof) another interesting consequence of the finiteness of $\bar{\omega}(E)$. If $\bar{\omega}(E)$ is finite in the sense

$$\bar{\omega}(E) = \begin{cases} 0 & E \geqslant E_2 \\ \omega(E) & E \in (E_1, E_2), \\ 0 & E \leqslant E_1 \end{cases} \tag{3.23}$$

then the function $p(t)$ is shown to be "quantized:"

$$p(t) = \exp\left\{-\frac{i}{\hbar}\frac{E_1 + E_2}{2}t\right\} \sum_{n=-\infty}^{\infty} p\left(\frac{2\pi n\hbar}{E_2 - E_1}\right) \frac{\sin[(E_2 - E_1)t/2\hbar + n\pi]}{(E_2 - E_1)t/2\hbar + n\pi}, \tag{3.24}$$

that is, the value of the decay probability $L(t)$ is completely determined by its values at the points $t_n = 2\pi n\hbar/(E_2 - E_1)$.

4. INVESTIGATION OF THE "DISPERSION" RELATIONS

In the second section we obtained "dispersion" relations, in particular, Eq. (2.16), which relate $M(t)$ and $N(t)$, i.e., the modulus and phase of the function $p(t)$ by integral equations. If one were to succeed in solving these equations, then, knowing $M(t)$ (from an experimental determination of the decay), we would be able to compute $N(t)$, and in the same way we could, on the basis of (2.3), determine the energy distribution density $\bar{\omega}(E)$ of the decaying system, without making any assumptions as to its nature,, i.e., as to the Hamiltonian operator. In this case we could speak of the following inverse problem: knowing the decay law $L(t)$, to determine analytically the energy distribution density $\bar{\omega}(E)$ of the decaying system.

With the aim of solving this inverse problem, let us study the dispersion relations (2.16). We consider the following function:

$$g(t) \equiv \log p_1(t) = \log M(t) - iN(t). \tag{4.1}$$

According to what was pointed out earlier, $p_1(t)$ is analytic in the upper half plane $\operatorname{Im} t > 0$, and consequently it can have only isolated poles in the upper half plane.

Assuming that all the poles of the function $p_1(t)$ lie in a semicircle of finite radius, on the basis of the theorem on the number of poles of an analytic function, we get[8]

$$N(\infty) = n\pi, \tag{4.2}$$

where n is the number of poles.

Initially, let us assume that $p_1(t)$, which is analytic in the upper half plane, has no poles there. Then $g(t)$ will also be an analytic function in the upper half plane $\operatorname{Im} t > 0$. However, it is not possible to apply Cauchy's formula to $g(t)$ directly, since $p_1(t) \to 0$, and consequently $g(t) \to \infty$ for $t \to \infty$. However, on the basis of the criterion of physical feasibility and (4.2), it is easy to estimate the order of magnitude of $g(t)$. Actually, from (3.3), (3.7), and (4.2),

$$g(t) \sim t^q, \quad t \to \infty; \quad q < 1. \tag{4.3}$$

Consequently, if we consider the function

$$g_n(t) \equiv t^{-n} g(t) = t^{-n} \log M(t) - it^{-n} N(t), \quad n \geqslant 2, \tag{4.4}$$

then, being an analytic function for $\operatorname{Im} t > 0$, it satisfies at the same time the conditions of applicability of the Cauchy formula, because $g_n(t)$ approaches zero sufficiently rapidly for $t \to \infty$.

Making use of the more general Cauchy formula in that case in which the analytic function $g_n(t)$

has poles of n-th order*

$$\frac{t^n}{i\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{g(t')}{t'^n(t'-t)}\,dt'=g(t)-g(0)-\frac{t}{1!}\frac{dg(0)}{dt}-\ldots-\frac{t^{n-1}}{(n-1)!}\frac{d^{n-1}g(0)}{dt^{n-1}}, \tag{4.5}$$

and taking it into account that

$$\mathrm{Re}\,\frac{d^{2m+1}g(0)}{dt^{2m+1}}=0;\ \mathrm{Im}\,\frac{d^{2m}g(0)}{dt^{2m}}=0,\ g(0)=0, \tag{4.6}$$

we get [on the basis of (4.5)] the following "dispersion" relations:

$$\begin{aligned}\mathrm{Re}\,g(t)&=\frac{t^2}{2!}\frac{d^2g(0)}{dt^2}+\cdots+\frac{t^{2m}}{(2m)!}\frac{d^{2m}g(0)}{dt^{2m}}+\frac{t^{2m+1}}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\mathrm{Im}\,g(t')}{t'^{2m+1}(t'-t)}\,dt',\\ \mathrm{Im}\,g(t)&=\frac{t}{1!}\frac{dg(0)}{dt}+\cdots+\frac{t^{2m-1}}{(2m-1)!}\frac{d^{2m-1}g(0)}{dt^{2m-1}}-\frac{t^{2m+1}}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\mathrm{Re}\,g(t')}{t'^{2m+1}(t'-t)}\,dt',\end{aligned} \tag{4.7}$$

or

$$\begin{aligned}\mathrm{Re}\,g(t)&=\frac{t^2}{2!}\frac{d^2g(0)}{dt^2}+\cdots+\frac{t^{2m-2}}{(2m-2)!}\frac{d^{2m-2}g(0)}{dt^{2m-2}}+\frac{t^{2m}}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\mathrm{Im}\,g(t')}{t'^{2m}(t'-t)}\,dt',\\ \mathrm{Im}\,g(t)&=\frac{t}{1!}\frac{dg(0)}{dt}+\cdots+\frac{t^{2m-1}}{(2m-1)!}\frac{d^{2m-1}g(0)}{dt^{2m-1}}-\frac{t^{2m}}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\mathrm{Re}\,g(t')}{t'^{2m}(t'-t)}\,dt'.\end{aligned} \tag{4.8}$$

We note that the terms outside the integral sign in these "dispersion" relations contain the moments the distribution density $\overline{\omega}(E)$:

$$\omega_k\equiv\int\limits_0^{\infty}E^k\overline{\omega}(E)\,dE. \tag{4.9}$$

In the special case when $m = 1$, i.e., $n = 2$, we get, after simple transformations, the following "dispersion" relations:

$$\log M(t)=-\frac{2t^2}{\pi}\,\mathrm{P}\int\limits_0^{\infty}\frac{1}{t'^2-t^2}\left\{\frac{N(t')}{t'}-\frac{N(t)}{t}\right\}dt',\quad N(t)=-t\,\frac{dp_1(0)}{dt}-\frac{2t^3}{\pi}\,\mathrm{P}\int\limits_0^{\infty}\frac{1}{t'^2-t^2}\left\{\frac{\log M(t')}{t'^2}-\frac{\log M(t)}{t^2}\right\}dt', \tag{4.10}$$

which permit us to determine $M(t)$ uniquely from the known values of $N(t')$; to determine $N(t)$ from known values of $M(t')$, we must also know $dp(0)/dt$, i.e., the actual mean value of the energy.

To obtain "dispersion" relations which permit us to compute $N(t)$ from known values of $M(t')$, we consider the function

$$g_-(t)\equiv g(t)-g(\infty). \tag{4.11}$$

Making use of (4.2), we then get the desired "dispersion" relation on the basis of the Cauchy formula (2.6

$$N(t)=\frac{2t}{\pi}\,\mathrm{P}\int\limits_0^{\infty}\frac{\log M(t')-\log M(t)}{t'^2-t^2}\,dt'. \tag{4.12}$$

It was assumed above that $p_1(t)$ has no poles in the upper half plane. In thise case we can determine $N(t)$ uniquely in terms of the known values of $M(t')$ and solve the inverse problem. If $p_1(t)$ has a pole in the half plane $\mathrm{Im}\,t > 0$, then the connection between $M(t)$ and $N(t)$ becomes ambiguous, but the resulting arbitrariness can be exactly identified.

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*The symbol P in (4.5) and below denotes an integral in the sense of the principal value both at the singular points $t' = t$ and at $t' = 0$.

Let $p_1(t)$ have a simple pole in the half plane $\operatorname{Im} t > 0$, say at $t_n = \alpha_n + i\beta_n$; $\beta_n > 0$, i.e.,

$$p_1(t) = \tilde{p}_1(t)(t - \alpha_n - i\beta_n), \tag{4.13}$$

where $\tilde{p}(t)$ is already an analytic function in the upper half plane $\operatorname{Im} t > 0$, having no poles there. We consider the function

$$\bar{p}_1(t) \equiv p_1(t)(t - \alpha_n + i\beta_n)/(t - \alpha_n - i\beta_n). \tag{4.14}$$

It is evident from (4.14) that the function $\bar{p}_1(t)$, which is analytic in the upper half plane, has no poles there (its poles are in the lower half plane), and has the same modulus as the original function $p(t)$. Applying the dispersion relation (4.12) to $p(t)$, we see that $N(t)$ is given by the following expression:

$$N(t) = \frac{2(t - \alpha_n)\beta_n}{(t - \alpha_n)^2 - \beta_n^2} + \frac{2t}{\pi} \mathrm{P} \int\limits_0^\infty \frac{\log M(t') - \log M(t)}{t'^2 - t^2}\, dt', \tag{4.15}$$

because

$$\arg \bar{p}_1(t) = \arg \bar{p}_1(t) - \arg \frac{t - \alpha_n + i\beta_n}{t - \alpha_n - i\beta_n}. \tag{4.16}$$

Thus, finally, knowing the modulus of the function $p(t) - M(t)$ [which is connected with the experimentally determined decay law $L(t) = M^2(t)$], and the poles of the function $p_1(t)$ in the upper half plane $\operatorname{Im} t > 0$, we can uniquely determine the phase of the function $p(t) - M(t)$ and also, from (2.3), we can determine analytically the energy distribution function $\overline{\omega}(E)$.

Some remarks relative to the experimental proof. As is easily understood from the conclusions reached above, it is necessary to observe the decay, beginning with the initial instant of time $t = 0$. For this purpose we could use an experiment with the formation of an artificial, short-lived isotope, fixing the initial moment of time $t = 0$ by having a "pulse" of particles bombard a stable nucleus, a product of which is an artificial short-lived isotope. In this same experiment, we could observe the departure of the decay law from the exponential.

In conclusion, I express my gratitude to V. A. Fock for his discussion of the work and for valuable comments, and also to the colleagues of the seminar of the Department of Theoretical Physics, Leningrad State University for discussion of the work.

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THE INNER BREMSSTRAHLUNG OF A POLARIZED MU MESON AND THE NONCONSERVATION OF PARITY

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On the hypothesis of the conservation of the combined parity, formulas are obtained for the angular and energy distributions of the inner bremsstrahlung accompanying the decay of a polarized μ meson.

FOR an unpolarized μ meson decaying into a neutrino, an antineutrino, and an electron, the bremsstrahlung has been calculated by Lenard[1] and Skorniakov.[2] It has now become evident[3,4] that parity is not conserved in decay interactions. In this connection it has been pointed out[5] that the neutrino can be a longitudinal particle.* This hypothesis can be regarded as experimentally confirmed.[6] Because of the longitudinal character of the neutrino, only the vector and pseudovector types of coupling take part in the interaction for the decay of the μ meson.[5]

In connection with the nonconservation of parity in the weak interactions, the question arises of the asymmetry of the angular distribution of the inner bremsstrahlung of a polarized μ meson. That such an asymmetry must exist can be seen just from the fact that the radiation is mainly in the direction of motion of the electron, and the electrons have a preferred direction of emission in the decay.

We use the method for calculating the decay probability, developed by Lenard,[1] and the method of calculation by means of polarization projection operators, developed by Tolhoek, Fano, and Michel.[7]

The probability of radiative decay in a range of momenta of the electron and the quantum is[1,2]

$$\mathfrak{W}(\mathbf{p},\mathbf{K})\,d^3K\,d^3p=(2\pi)^{-8}\iint d^3k\,d^3k'\frac{1}{2}\sum_s\sum_w|(f\,|\,R\,|\,i)|^2\,\delta(P-K-k-k'-p)\,d^3K\,d^3p. \tag{1}$$

Here and below we have used the notations of Ref. 1: M and m are the masses of the μ meson and electron; ϵ and W are the energies of the electron and photon; P, p, K, k′, and k are the four-momenta of the μ meson, electron, photon, neutrino, and antineutrino, and **p** and **K** are three-dimensional momenta. We set $\hbar = c = 1$ and $e^2/4\pi = 1/137$.

The matrix element for the transition is:

$$(f\,|\,R\,|\,i)=eg\,[\bar{\nu}(k)(1+\gamma_5)\gamma_\sigma(1-\gamma_5)\nu(k')]\left\{u(p)\left[\frac{w}{(2W)^{1/2}}\frac{\hat{q}+im}{(qq)+m^2}\gamma_\sigma+\gamma_\sigma\frac{\hat{Q}+iM}{(QQ)+M^2}\frac{\hat{w}}{(2W)^{1/2}}\right](1+\lambda\gamma_5)\,U(P)\right\}. \tag{2}$$

In Eq. (2) the nonconservation of parity has already been taken into account. u(p) and U(P) are the wave functions of the electron and μ meson; $q = p + K$ and $Q = P - K$ are the momenta of the electron and μ meson in the intermediate state. The probability is found by summing over the spin states of the electron and the polarizations of the quantum:

$$\mathfrak{W}(\mathbf{p},\mathbf{K})=\frac{1}{4}\frac{e^2g^2}{(2\pi)^8\,\varepsilon W}F;\quad F=\frac{\pi}{12}(GG)\,\mathrm{Sp}\left\{\hat{p}\left[\frac{\hat{w}}{pK}(\hat{p}+\hat{K})\gamma_\sigma-\gamma_\sigma(\hat{P}-\hat{K}+iM)\frac{\hat{w}}{PK}\right](1+\lambda\gamma_5)(1+\beta)(1+\mathbf{J}\gamma_5\beta\gamma)\right.$$
$$\left.\times\left[\gamma_\sigma(\hat{p}+\hat{K})\frac{w}{pK}-\frac{\hat{w}}{PK}(\hat{P}-\hat{K}-iM)\right](1+\lambda\gamma_5)\right\}\frac{\pi}{12}\mathrm{Sp}\left\{\hat{p}\left[\frac{\hat{w}}{pK}(\hat{p}+\hat{K})\hat{G}-\hat{G}(\hat{P}-\hat{K}+iM)\frac{\hat{w}}{PK}\right](1+\lambda\gamma_5)(1+\beta)\right.$$
$$\left.\times(1+\mathbf{J}\gamma_5\beta\gamma)\left[\hat{G}(\hat{p}+\hat{K})\frac{\hat{w}}{pK}-\frac{\hat{w}}{PK}(\hat{P}-\hat{K}-iM)(1+\lambda\gamma_5)\right\},$$

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*Note (October 9, 1957): In connection with the latest data on the neutrino and antineutrino, it must be pointed out that since in the problem considered here the essential point is just the existence of the vector and pseudovector types of interaction, which can be regarded as experimentally established[7] by the measurements of the energy spectrum of the electrons, the hypothesis of the longitudinal character of the neutrino leads only to a renormalization of the interaction constant (the renormalization constant being 2).

$$G \equiv P - p - K; \quad \hat{P} \equiv (P\gamma) \text{ etc.} \tag{3}$$

Here **J** is a unit vector giving the direction of the spin of the μ meson. The calculations give a formula of the following form:

$$F = \frac{\pi}{3}\{f_1 + (\mathbf{p}\mathbf{w}_1)^2 f_2 + (\mathbf{pJ})(\mathbf{p}\mathbf{w}_1)^2 f_3 + (\mathbf{pJ}) f_4 + (\mathbf{KJ}) f_5 + (\mathbf{p}\mathbf{w}_1)(\mathbf{p}, [\mathbf{J}\times\mathbf{w}_1]) f_6 + (\mathbf{p}\mathbf{w}_1)^2 (\mathbf{KJ}) f_7 + (\mathbf{J}\mathbf{w}_1)(\mathbf{p}\mathbf{w}_1) f_8. \tag{4}$$

Here

$$\mathbf{w}_1 = \mathbf{K}\times[\mathbf{p}\times\mathbf{K}]/|\mathbf{K}\times[\mathbf{p}\times\mathbf{K}]|, \qquad f_i = f_i(\varepsilon, M, W, (\mathbf{pK})).$$

The values of the functions f_i and the limiting cases for them are given in the appendix.

The terms in Eq. (1b) of the appendix that contain in their denominators the quantity $(pK - \epsilon W)$ to only the zeroth or first degree are small, since in them integration over the angles does not give rise to the factor $\ln(2\epsilon/m)^2$. After neglecting such terms one gets the formula

$$\mathfrak{W}(\mathbf{K}, \mathbf{p})\, d^3p\, d^3K = \frac{1}{2}\frac{e^2}{(2\pi)^3}\frac{(\mathbf{p}\mathbf{w}_1)^2 W^2}{(\mathbf{pK} - \varepsilon W)^2}\mathfrak{W}(\mathbf{p})\, d^3p d^3K. \tag{5}$$

Here $\mathfrak{W}(\mathbf{p})$ is the probability for the emission of the electron into a given range of angles and energy, as given by Landau's formula.[5] Equation (5) has just the same sort of form as Lenard's formula,[1] but now $\mathfrak{W}(\mathbf{p})$ stands for a different expression.

The precision of Eq. (5) is given by the ratio $1 : \ln(2\epsilon/m)^2$, or about $1/10$.

In order to find the total probability for emitting a quantum one must integrate Eq. (3) with allowance for (1b).

Since we are confining ourselves to just the region of small frequencies and thus the condition $W \leq M/2 - \epsilon$ is fulfilled everywhere except in a very small range of energies of the electron, we can ignore the fact that near the upper limit of the integration, given by $M/2$, not all angles between the momenta of the electron and the quantum are allowed. Integrating, we get

$$\mathfrak{W}(\mathbf{K})\, d^3K = \frac{e^2g^2\, dW}{24\,(2\pi)^6\, W}\, d\Omega_\gamma \left\{64\left[(1+\lambda^2)\left(\frac{\varepsilon_m^5}{2}\ln\frac{2\varepsilon_m}{m} - \frac{5}{24}\varepsilon_m^5\right) - 2\lambda\frac{\mathbf{KJ}}{W}\varepsilon_m^5\left(\frac{1}{6}\ln\frac{2\varepsilon_m}{m} - \frac{1}{72}\right)\right]\right\}. \tag{6}$$

The ratio of the numbers of quanta emitted at small frequencies in directions with and against the spin (the numbers of quanta integrated over the hemispheres) is 1 : 1.5 for $\lambda = 1$. Let us calculate the probability of emission of a quantum with energy larger than a certain value W_0. We confine ourselves to terms containing the factor $\ln(2\epsilon/m)^2$, and thus get an accuracy of about 10%.

In integrating the expression (3) over the angles, using Eq. (1c) one must note the fact that at large energies not all angles between the momenta of the electron and the quantum are allowed; the limits of integration are therefore as follows: $-1 \leq \cos(\mathbf{Kp}) \leq 1$ for $W \leq M/2 - \epsilon$ and $-1 \leq \cos(\mathbf{Kp}) \leq a$ for $W \geq M/2 - \epsilon$. Here $a = M^2/2pW - M/W - M/p + 1$.

As the result of the integration we get

$$\begin{aligned}\mathfrak{W}\left(\frac{\mathbf{K}}{W}, W_0\right) d\Omega_\gamma = \frac{e^2g^2}{(2\pi)^6\, 24}\Bigg[\Bigg\{(3\varepsilon M^2 - 4\varepsilon^2 M)\int\limits_{M/2-\varepsilon}^{M/2}\frac{AC\ln W}{W(BW+C)}\, dW - (\varepsilon - M)(M - 6\varepsilon) M \ln\frac{(2\varepsilon/m)^2}{(2\varepsilon/M)+1} \\ + \Big(-(3\varepsilon M^2 - 4\varepsilon^2 M)\ln W_0 - (3M^2 - 8M\varepsilon) W_0 + 2MW_0^2\Big)\ln\left(\frac{2\varepsilon}{m}\right)^2\Bigg](1+\lambda^2) + 2\lambda\,(\mathbf{pJ})\Bigg[\Big(-12M\varepsilon\ln W_0 - 46MW_0 \\ - \frac{3M^2}{\varepsilon^2}W_0^2 - \frac{8}{3}\frac{M}{\varepsilon^2}W_0^3\Big) 2\ln\frac{2\varepsilon}{m} + 12M\varepsilon\int\limits_{\frac{M}{2}-\varepsilon}^{M/2}\frac{AC\ln W}{W(BW+C)}\, dW - \left(\frac{133}{3}M\varepsilon - 41M^2 - \frac{2M^2}{\varepsilon} + \frac{5M^4}{\varepsilon^2}\right)\ln\frac{(2\varepsilon/m)^2}{(2\varepsilon/M)+1}\Bigg]\Bigg\} d\Omega_\gamma.\end{aligned} \tag{7}$$

Here

$$A \equiv 1 + \varepsilon/|\mathbf{p}|; \qquad B \equiv (2\varepsilon - 2M)/|\mathbf{p}|; \qquad C \equiv (M - 2\varepsilon) M/|\mathbf{p}|.$$

The total energy radiated into the given range of angles, as obtained in the same approximation as the preceding formula, is given by

$$\frac{e^2g^2}{80\,(2\pi)^6} M^6 [(1+\lambda^2) - 2\lambda \cos(\mathbf{KJ})]. \tag{8}$$

We observe that the radiation is symmetric with respect to the plane of **p** and **J**. I thank L. B. Okun' for the suggestion of the problem and K. A. Ter-Martirosian and I. S. Tsukerman for discussions.

APPENDIX

Exact values of the functions f_i:

$$f_1 = -\{3M^2W/(pK) - 4MW^2/(pK) - 4M\varepsilon W/(pK) + 4W + 3(pK)/W$$
$$- 4(pK)/M - 4\varepsilon(pK)/M^2 + 4(pK)^2/M^2W + 2M\}(1+\lambda^2),$$

$$f_2 = [(1+\lambda^2)/M(pK)^2]\{3\varepsilon M^3 + 3M^3W - 3(pK)M^2 - 4\varepsilon^2M^2 - 8M^2\varepsilon W - 4M^2W^2 + 8MW(pK) + 8\varepsilon M(pK) - 4(pK)^2\},$$

$$f_3 = -2\lambda(pK)^{-2}\{-4M^2 + 16M\varepsilon + 16MW\},$$

$$f_4 = 2\lambda\{-3M^2/W^2 + 8M\varepsilon/W^2 + 10M/W - 2pK/W^2 - 8\varepsilon/W + 2(pK)/MW + 4MW/pK\},$$

$$f_5 = 2\lambda\{-6\varepsilon M/W^2 + 12\varepsilon^2/W^2 - 4\varepsilon(pK)/MW^2 + 6M^2/(pK) - 4M\varepsilon/(pK) - 8MW/(pK) + 20 - 2(pK)/W^2$$
$$+ 4\varepsilon/W + 18(pK)/MW - 4(pK)^2/MW^2 - 6M^2\varepsilon/(pK)W + 14M\varepsilon^2/(pK)W + 2M/W\},$$

$$f_6 = -(1+\lambda^2)\{-8M/(pK) + 24\varepsilon/(pK) + 24W/(pK) + 4/W\},$$

$$f_7 = -2\lambda\left\{\frac{1}{(pK)W}(8\varepsilon - 4M) + \frac{24}{(pK)} - \frac{8}{MW} + \frac{4M^2}{(pK)^2}\right\},$$

$$f_8 = 2\lambda\left\{\frac{8M\varepsilon - 6M^2 - 16pK}{W^2} + \frac{4M}{(pK)}(M - 2W) - \frac{2M^2\varepsilon}{W(pK)} - \frac{14M}{W} + \frac{24\varepsilon}{W} + 54 - \frac{2M\varepsilon^2}{W(pK)} + \frac{2\varepsilon M}{(pK)} - \frac{24(pK)}{MW}\right\}. \tag{1a}$$

Here $(pK) \equiv pK - \epsilon W$.

If we confine ourselves just to the case of small frequencies, i.e., if $W \ll \epsilon$, then the functions f_i go over into the following:

$$f_1 = f_5 = f_6 = f_7 = 0; \quad f_2 = \frac{1}{M(\mathbf{pK} - \varepsilon M)^2}\{3\varepsilon M^3 - 4\varepsilon^2M^2\}(1+\lambda^2), \quad f_3 = -2\lambda(16M\varepsilon - 4M^2)/(\mathbf{pK} - \varepsilon W)^2;$$
$$f_4 = 2\lambda(8M\varepsilon - 3M^2)/W^2, \quad f_8 = 2\lambda\{(-6M^2 + 8M\varepsilon)/W^2 - (2M^2\varepsilon + 2M\varepsilon^2)/W(\mathbf{pK} - \varepsilon W)\}. \tag{1b}$$

If we include only those terms that give $\ln(2\epsilon/m)^2$ on integration, then the functions f_i go over into the following:

$$f_2 = \frac{(1+\lambda^2)}{M(\mathbf{pK} - \varepsilon W)^2}(3\varepsilon M^2 - 4\varepsilon^2M^2 + 3M^3W - 8M^2\varepsilon W - 4M^2W^2),$$
$$f_3 = -2\lambda\{-4M^2 + 16M\varepsilon + 16MW\}/(\mathbf{pK} - \varepsilon W)^2, \quad f_7 = -2\lambda\, 4M^2/(\mathbf{pK} - \varepsilon W)^2. \tag{1c}$$

Note added in proof (November 15, 1957). While the present paper was in press, papers were published by Kinoshita and Sirlin, in which effects of radiative corrections and inner bremsstrahlung on the asymmetry of the positrons are calculated.

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EQUATIONS OF MOTION FOR ROTATING MASSES IN THE GENERAL THEORY OF RELATIVITY

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The systems to be considered consist of n spherically symmetrical objects of infinitesimal size, which possess angular momentum and which interact only through gravitational forces. For such systems, it is relatively simple to derive the relativistic equations of translation and rotation from the Einstein gravitational equations. This derivation uses L. Infeld's idea of introducing Dirac delta-functions into the energy-momentum tensor.

THE motions of masses in their own gravitational fields is a well-known and important problem. In particular, the important question: "Are the equations of motion for the masses contained in the equations for their gravitational field?" was answered affirmatively by Einstein.[1,2] Methods for deriving detailed equations of motion from the field equations of general relativity theory have recently been developed along two lines, represented on the one hand by the fundamental work of Einstein, Infeld, and their co-workers[3-5] and on the other by Fock and his co-workers[6,7] and Papapetrou.[8] Fock[9] uses his approximational method to derive non-Newtonian equations and their integrals for the translation of finite rotating masses.

The aim of the present paper is to derive non-Newtonian equations for the translation and rotation of infinitesimally small rotating, spherically symmetrical objects, on the basis of the approximate method developed by Einstein's school. The starting point is a paper by Infeld,[10] who gives a relatively simple derivation of the non-Newtonian equations of translational motion for a non-rotating object represented by a singularity in the gravitational field. This derivation is so simple that Infeld, abandoning the tradition of the Einstein school of deriving the equations of motion from the free-space field equations, makes use, like Papapetrou,[8] of the vanishing of the divergence of the energy-momentum tensor, a result of Einstein's gravitational equations. At the same time he introduces a novelty in the form of Dirac delta-functions, corresponding to the infinitesimal spherically-symmetrical particles mentioned above, as part of the energy-momentum tensor. By "smearing out" the delta-functions he arrives at a continuous distribution of matter, e.g. at the spherically symmmetrical object of finite dimensions which is used repeatedly in his paper. Thus, this paper by Infeld,[10] serves in many ways as a link between the methods of Einstein and Fock.

1. As already mentioned, the present paper will make use of Infeld's method[10] for deriving the equations of motion. In this method we start essentially from the known equations for the graviational field

$$R^{\alpha\beta} - {}^1/_2\, g^{\alpha\beta} R = -8\pi T^{\alpha\beta}, \tag{1.1}$$

where Newton's constant γ and the speed of light c have been set equal to unity, and the other quantities have their usual meaning.* Let us consider the motion of a system of n gravitating point particles. We assume here that the world lines of the particles do not intersect. Then Infeld takes as $\tilde{T}^{\alpha\beta}$, for a system of n particles without angular momentum, the expression

$$\tilde{T}^{\alpha\beta} \equiv \sqrt{-g}\, T^{\alpha\beta} = \sum_a \tilde{T}_a^{\alpha\beta}, \qquad \tilde{T}_a^{00} = m_a \delta_a, \quad \tilde{T}_a^{0l} = m_a \frac{da^l}{dx^0}\delta_a, \quad \tilde{T}_a^{ik} = m_a \frac{da^i}{dx^0}\frac{da^k}{dx^0}\delta_a. \tag{1.2}$$

Here the sum is to be taken over all the n particles of the system, which are distinguished by the small letters a, b, ...; there is no summation over repeated subscripts a, b, ...; m_a is a function only of the time, x^0; a^i are the coordinates of the a-th particle; $\delta_a = \delta(x^i - a^i)$ is the usual three-dimensional delta-function, referring to particle a. Expression (1.2) for $\widetilde{T}^{\alpha\beta}$ is derived from the energy-momentum tensor density for a dust cloud

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*Greek indices α, β, ... take the values 0, 1, 2, and 3, referring to both space and time coordinates. Latin indices i, k, ℓ, ... take the values 1, 2, and 3, referring only to the spatial coordinates.

$$\widetilde{T}^{\alpha\beta} = \sqrt{-g}\,\rho U^{\alpha} U^{\beta}, \quad U^{\alpha} = dx^{\alpha}/ds, \tag{1.3}$$

by substituting $\rho = \sum \rho_a = \sum m'_a \delta_a$ (i.e., each spherically symmetrical cloud of material, described by a function ρ_a, is replaced by a material point having the same mass and located at its center of gravity). Therefore, $m_a = \sqrt{-g}(dx^0/ds)^2 m'_a$. Integrating the relation

$$\widetilde{T}^{\alpha\beta}_{;\beta} \equiv \sqrt{-g}\,T^{\alpha\beta}_{;\beta} \equiv \widetilde{T}^{\alpha\beta}_{,\beta} + \Gamma^{\alpha}_{\beta\sigma}\widetilde{T}^{\beta\sigma} = 0, \tag{1.4}$$

which comes from (1.1), over a three-dimensional volume which contains the a-th particle but no others, we obtain the 4n equations

$$\int\limits_{V_a} \widetilde{T}^{\alpha\beta}_{;\beta} dV = 0, \tag{1.5}$$

from which the equations of translational motion can be derived.*

Note that although the equations in (1.5) are not covariant in the usual sense (this is not a four-vector) they are still "covariant" in the sense that they preserve their form and symmetry in any system of coordinates which permits an expansion in powers of λ.† This is a reasonable conclusion, in view of the presence of delta-functions in $\widetilde{T}^{\alpha\beta}$. For, suppose the covariant expression

$$\widetilde{T}^{\alpha\beta}_{;\beta} d\Omega \equiv \widetilde{T}^{\alpha\beta}_{;\beta} dV dx^0 = 0$$

is integrated over an extended four-volume Ω_a containing the corresponding a-th particle at the world point x^{α}_a. Then it is well known that upon going to the limit $\Omega_a \to 0$ we again obtain a covariant expression. Since $\widetilde{T}^{\alpha\beta}$ contains delta-functions, we can make use of a three-dimensional volume V_a with extended boundaries. The theorem of the mean can be applied to the integral over dx^0. All this leads to

$$\int\limits_{\Omega_a} \widetilde{T}^{\alpha\beta}_{;\beta} d\Omega = \int\limits_{\Delta x^0} dx^0 \int\limits_{V_a} \widetilde{T}^{\alpha\beta}_{;\beta} dV = \Delta x^0 \Big[\int\limits_{V_a} \widetilde{T}^{\alpha\beta}_{;\beta} dV + O(\Delta x^0) \Big] = 0,$$

where $O(\Delta x^0) \to 0$ as $\Delta x^0 \to 0$. Upon dividing the last equation by Δx^0 and then going to the limit as $\Delta x^0 \to 0$ (that is to say, in the limit as $\Omega_a \to 0$) we obtain Eqs. (1.5).‡

2. The equations of translational motion can be derived from (1.5) with the aid of an approximate method. The particular method which is to be used will depend fundamentally on the choice of an expansion parameter λ, chosen in the following way. Let $\varphi(x^{\alpha})$ be any of the functions occurring in (1.1). Assume that, in the system of units where the speed of light c is unity, $\partial\varphi/\partial x^0 \ll \partial\varphi/\partial x^i$, i.e., φ varies rapidly with position in space, but changes slowly in time. (This assumption implies that the speed of the moving objects which create the gravitational field is small compared with the speed of light.) Hence we may say that $\partial\varphi/\partial x^0$ is an order of magnitude smaller than $\partial\varphi/\partial x^i$. In order to express this analytically, let us introduce the auxiliary time variable

$$\tau = \lambda x^0, \tag{2.1}$$

where the parameter λ is chosen so that $\varphi_{,i} \equiv \partial\varphi/\partial x^i$ and $\varphi_{,0} \equiv \partial\varphi/\partial\tau$ are of the same order of smallness. We now expand all the functions occurring in (1.1) in powers of λ in the following way. From

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*Repeated indices are to be summed, as usual. Semicolons represent covariant differentiation, and commas represent the usual partial differentiation. g is the determinant consisting of the components of the covariant metric tensor $g_{\alpha\beta}$.

†Such coordinate systems are essentially restricted to those in which a unique solution of the equations of motion exists. (They are Galilean in the zeroth approximation and harmonic in the first.) The coordinate conditions which arise in this problem have been discussed many times in the literature,[5,9–14] and since they have no direct bearing here and require a special treatment, they will not be considered now.

‡Another argument for the "covariance" of (1.5) is the fact that the equations of motion derived from them by Infeld,[10] for non-rotating particles, agree exactly with the equations derived earlier.[3,5,7,8]

Newton's law of universal gravitation we deduce that the expansion of m_a must begin with the second degree, since the acceleration $d^2a^i/dx^{02} = \lambda^2 d^2a^i/d\tau^2 \equiv \lambda^2 \ddot{a}^i$ is of the second degree. Therefore, according to Infeld,[10] we assume the expansion of m_a to be of the form

$$m_a = \lambda^2 \underset{2}{m_a} + \lambda^4 \underset{4}{m_a} + \cdots, \tag{2.2}$$

which determines the expansions of all the other functions through the use of (1.1). For example,

$$\begin{aligned} \widetilde{T}^{00} &= \lambda^2 \underset{2}{\widetilde{T}^{00}} + \lambda^4 \underset{4}{\widetilde{T}^{00}} + \ldots, \\ \widetilde{T}^{0i} &= \lambda^3 \underset{3}{\widetilde{T}^{0i}} + \lambda^5 \underset{5}{\widetilde{T}^{0i}} + \ldots, \\ \widetilde{T}^{ik} &= \lambda^4 \underset{4}{\widetilde{T}^{ik}} + \lambda^6 \underset{6}{\widetilde{T}^{ik}} + \ldots; \end{aligned} \tag{2.3}$$

$$\begin{aligned} g_{00} &= 1 + \lambda^2 \underset{2}{h_{00}} + \lambda^4 \underset{4}{h_{00}} + \ldots, \\ g_{0i} &= \qquad \lambda^3 \underset{3}{h_{0i}} + \ldots, \\ g_{ik} &= -\delta_{ik} + \lambda^2 \underset{2}{h_{ik}} + \ldots. \end{aligned} \tag{2.4}$$

The expansion of $g^{\alpha\beta}$ is found from the conditions $g_{\alpha\beta}g^{\beta\sigma} = \delta^\sigma_\alpha$; as shown by Infeld,[15] successive terms in these series differ by two orders.

3. We now have to choose $\widetilde{T}^{\alpha\beta}$ for the case of rotating particles. For this purpose, consider a number of objects a of finite size, whose centers of gravity move in a Euclidean space with velocities $\mathbf{v}_a(\tau)$ while at the same time they rotate rigidly around their centers of gravity with angular velocities $\boldsymbol{\omega}_a(\tau)$. Then the velocity of any point in an object is given by the sum $\mathbf{v}'_a = \mathbf{v}_a + \boldsymbol{\omega}_a \times \mathbf{r}_a$, where $\mathbf{r}_a \equiv \mathbf{r} - \mathbf{a}$ is the radius vector from the object's center of gravity to the point in question. The following two questions now arise: (1) To what does the additional term $\boldsymbol{\omega}_a \times \mathbf{r}_a$ reduce when the object shrinks to a point at its center of gravity; and (2) in what way is this to be included in $\widetilde{T}^{\alpha\beta}$?

Using an analogy from the hydrodynamics of vortical motion and from the determination of the vector field in classical field theory[16] (particles with spin correspond to our rotating, infinitesimal, spherically symmetrical objects) we assume that $\boldsymbol{\omega}_a \times \mathbf{r}_a$ can be replaced by

$$^1/_2\,[\sigma_a \times \nabla\delta_a], \tag{3.1}$$

where $\sigma_a = \sigma_a(\tau)$ is a pseudovector function of τ, and $\nabla \equiv \mathbf{e}_i\,\partial/\partial x^i$ operates on δ_a.

To find $\widetilde{T}^{\alpha\beta}$ we shall again start from (1.3), but this time the four-velocity U^α must include the additional effect of rotation, (3.1). To see how this additional effect can be introduced into U^α, consider an arbitrary object a of finite dimensions, the position of whose center of gravity is described by a point $M_a(x^\alpha_a)$ in a Riemann space-time with a metric determined by the system of n particles which we are considering. If a point $M(x^\alpha)$ is infinitely close to the point M_a, i.e., $x^\alpha = x^\alpha_a + \Delta x^\alpha$, then its four-velocity is (noting that $x^i_a = a^i$, and separating the space and time components)

$$U = \left\{\frac{dx^i}{ds};\ \frac{dx^0}{ds}\right\} = \left\{\frac{dx^0}{ds}\frac{dx^0_a}{dx^0}\left(\frac{da^i}{dx^0_a} + \frac{d\Delta x^i}{dx^0_a}\right);\ \frac{dx^0}{ds}\right\}, \tag{3.2}$$

where the quantity $d\Delta x^i/dx^0_a$ describes the velocity of the point x^i relative to the point $x^i_a = a^i$, measured at the time x^0_a. Upon going to the limit (shrinking the object to a point at its center of gravity) this leads to (3.1), as expected. As for the ratio dx^0_a/dx^0, it is easy to expand it in λ:

$$q_a \equiv dx^0_a\,/\,dx^0 = d\tau_a\,/\,d\tau = 1 + \lambda^4 \underset{4}{q_a} + \ldots \tag{3.3}$$

(the equation $U^\alpha U_\alpha = 1$ must be solved for dx^0_a/dx^0 by approximations or by Landau's Eq. (82.7). In the absence of rotation, this reduces to unity. Substituting (3.2) into (1.3), taking account of (3.1) and (3.3), and introducing the delta-functions, we obtain

$$\widetilde{T}^{\alpha\beta} = \sum_a \widetilde{T}^{\alpha\beta}_a,$$

$$\widetilde{T}^{00}_a = m_a\delta_a, \qquad \widetilde{T}^{0i}_a = m_a q_a\,(\lambda\dot{a}^i + {}^1/_2\,[\sigma_a \times \nabla]^i)\,\delta_a, \qquad \widetilde{T}^{ik}_a = m_a q_a^2\,(\lambda\dot{a}^i + {}^1/_2\,[\sigma_a \times \nabla]^i)\,(\lambda\dot{a}^k + {}^1/_2\,[\sigma_a \times \nabla]^k)\,\delta_a. \tag{3.4}$$

Here m_a has an expansion in the form of (2.2), but is not the same as the m_a in (1.2). The pseudovector σ_a has the expansion

$$\sigma_a = \lambda \underset{1}{\sigma}_a + \lambda^3 \underset{3}{\sigma}_a + \dots, \tag{3.5}$$

since we naturally require that the tensor density $\tilde{T}^{\alpha\beta}$ in (3.4) have an expansion of the form (2.3) in λ. $\dot{a}^i \equiv da^i/d\tau$, and ∇ operates on $\delta_a = \delta(x^i - a^i)$. When $\sigma_a = 0$, (3.4) reduces to (1.2). Notice also that in the following derivation of the equations of motion, in the Newtonian approximation, we make use of $\underset{5}{\tilde{T}}{}^{0i}$ and $\underset{6}{\tilde{T}}{}^{ik}$ but no terms of higher order, so that in (3.4) we may put $q_a = 1$.

Once $\tilde{T}^{\alpha\beta}$ has been found, we can obtain the equations of motion from (1.5). But now, in addition to the 4n unknown functions $m_a(\tau)$ and $a^i(\tau)$, which in the absence of rotation are completely determined by the 4n ordinary differential equations (1.5), another 3n unknown functions $\sigma_a^i(\tau)$ make their appearance. Equations (1.5) must therefore be supplemented by the 3n equations

$$\int\limits_{V_a} (x^i \tilde{T}^{k\alpha}_{;\alpha} - x^k \tilde{T}^{i\alpha}_{;\alpha})\, dV = 0, \tag{3.6}$$

from which the equations of rotational motion are to be derived, and which also arise from Einstein's gravitational equation (1.1). The derivation of eqation (3.6) is completely analogous to the derivation of (1.5) if we start with the covariant expression

$$(x^\alpha \tilde{T}^{\beta\nu}_{;\nu} - x^\beta \tilde{T}^{\alpha\nu}_{;\nu})\, d\Omega = 0.$$

4. The derivation of the equations of motion requires a knowledge of the metric, to the corresponding order of approximation, which means that $\underset{2}{h}_{00}$, $\underset{2}{h}_{ik}$, $\underset{3}{h}_{0i}$, and $\underset{4}{h}_{00}$ have to be determined from (1.1).

In our case the equations for determining $\underset{2}{h}_{00}$ and $\underset{2}{h}_{ik}$ are identical with the corresponding equations of Infeld,[10] and therefore they have the identical solution

$$\underset{2}{h}_{00} = -2 \sum_a \underset{2}{m}_a / r_a, \quad r_a \equiv |\mathbf{r} - \mathbf{a}|, \tag{4.1}$$

$$\underset{2}{h}_{ik} = \delta_{ik} \underset{2}{h}_{00}. \tag{4.2}$$

However, the equations which determine $\underset{3}{h}_{0i}$ are no longer the same as Infeld's, but are of the form

$$\underset{3}{h}_{0i,\,ss} - \underset{3}{h}_{0s,\,si} = -2 \underset{2}{h}_{00,\,\underset{1}{0}i} - 16\pi \sum_a \underset{2}{m}_a \dot{a}^i \delta_a - 8\pi \sum_a \underset{2}{m}_a [\underset{1}{\sigma}_a \times \nabla \delta_a]^i. \tag{4.3}$$

We write the solution to this in the form

$$\underset{3}{h}_{0i} = 4 \sum_a \frac{\underset{2}{m}_a \dot{a}^i}{r_a} - 2 \sum_a \underset{2}{m}_a \left[\underset{1}{\sigma}_a \times \nabla \frac{1}{r_a} \right]^i, \tag{4.4}$$

which differs from the corresponding expression in Infeld's paper[10] by the presence of the second term. Comparing the latter with (100-7) [sic!] in Ref. 14, we are led to an explanation of the physical significance of σ_a; this is nothing but the specific angular momentum for the a-th particle: $M_a = m_a \sigma_a$.

Knowing $\underset{2}{h}_{00}$, $\underset{2}{h}_{ik}$, and $\underset{3}{h}_{0i}$, we can derive the third, fourth, and fifth order equations of translational motion, as well as the fourth order rotational equations, from Eqs. (1.5) and (3.6). In the third order, the equations of translation (1.5) lead to the relation

$$\underset{2}{m}_a = \text{const}; \tag{4.5}$$

in the fourth order, to the Newtonian equations of motion

$$\underset{2}{m}_a \ddot{a}^i = \sum_{b \neq a} (\underset{2}{m}_a \underset{2}{m}_b / r_{ab})_{,\,a^i}; \tag{4.6}$$

and in the fifth order, to a value of $\underset{4}{m}_a$ identical with that of Infeld:[10]

$$\underset{4}{m}_a = {}^1\!/_2 \underset{2}{m}_a \dot{a}^s \dot{a}^s + \sum_{b \neq a} \underset{2}{m}_a \underset{2}{m}_b / r_{ab} + \underset{4}{C}_a, \tag{4.7}$$

where $\underset{4}{C}_a$ is a constant, and $r_{ab} \equiv |a - b|$. So far there has been no effect due to the rotation — all the integrals containing σ vanish — and our results (4.5) to (4.7) agree with Infeld's. The manipulations required to derive (4.1) to (4.7) are completely analogous to his[10] and need not be discussed in detail.

The equations of rotational motion (3.6) in the fourth (lowest) order, after a short, straightforward development, lead to the simple relations

$$\underset{3}{\dot{M}}{}_a^{ik} = 0, \quad \underset{3}{M}{}_a^{ik} = \text{const}; \text{ for } \underset{1}{\dot{\sigma}}{}_a = 0, \quad \underset{1}{\sigma}{}_a = \text{const}, \tag{4.8}$$

where

$$\underset{3}{M}{}_a^{ik} \equiv \delta_{ikl} \underset{2}{m}{}_a \underset{1}{\sigma}{}_a^l \tag{4.9}$$

($\delta_{ik\ell} = \pm 1$, $\delta_{123} = 1$ is a completely antisymmetric pseudotensor). Thus in this Newtonian approximation the angular momentum of each spherically symmetrical object is individually conserved, as expected.

5. Derivation of the sixth-order rotational equation of motion requires a knowledge of $\underset{4}{h}_{00}$. Infeld[10] carried out a detailed derivation of those terms in $\underset{4}{h}_{00}$ which affect the sixth-order equations of motion without rotation. Therefore in this paper we shall consider only the additional terms $\underset{4}{h}{}^*_{00}$ due to the rotation. By methods similar to Infeld's, we arrive at the following equations for $\underset{4}{h}{}^*_{00}$:

$$\underset{4}{\overset{\bullet}{h}}{}_{00,ss} = 8\pi \sum_a \underset{2}{m}{}_a \left(\dot{a}^s [\underset{1}{\sigma}{}_a \times \nabla \delta_a]^s + \frac{1}{4} [\underset{1}{\sigma}{}_a \times \nabla]^s [\underset{1}{\sigma}{}_a \times \nabla \delta_a]^s\right). \tag{5.1}$$

The solution of this takes the form

$$\underset{4}{\overset{\bullet}{h}}{}_{00} = 2 \sum_a \underset{2}{m}{}_a \left(\dot{a}^s \left[\underset{1}{\sigma}{}_a \times \nabla \frac{1}{r_a}\right]^s - \frac{1}{4} [\underset{1}{\sigma}{}_a \nabla]^s \left[\underset{1}{\sigma}{}_a \times \nabla \frac{1}{r_a}\right]^s\right). \tag{5.2}$$

Now by virtue of the linearity of the original differential equations, up to terms involved in the sixth-order equations of motion for the a-th object, $\underset{4}{h}_{00}$ may be expressed in the form

$$\underset{4}{h}_{00} \sim 2 \left(\sum_{b \neq a} \frac{\underset{2}{m}{}_b}{r_b}\right)^2 + \sum_{b \neq a} \left(- \underset{2}{m}{}_b \underset{11}{r}{}_{b,00} - 3\dot{b}^s \dot{b}^s \frac{\underset{2}{m}{}_b}{r_b} + 2 \frac{\underset{2}{m}{}_a \underset{2}{m}{}_b}{r_{ab} r_b}\right) + \sum_{b \neq a} \sum_{b \neq c\; c \neq a} \frac{\underset{2}{m}{}_b \underset{2}{m}{}_c}{r_{bc} r_c} + \underset{4}{\overset{\bullet}{h}}{}_{00}, \tag{5.3}$$

All the terms on the right hand side, with the exception of $\underset{4}{h}{}^*_{00}$, are already available. They describe the field for non-rotating particles to the indicated degree of approximation.

The sixth-order translational equations of motion (or the second approximation, according to V. A. Fock's terminology) for body a is given by the expression

$$\begin{aligned} \int_{V_a} (\underset{6}{\widetilde{T}}{}^{i\alpha}_{;\alpha})\, dV \equiv \int_{V_a} \Big[& \underset{5}{\widetilde{T}}{}^{i0}_{,\underset{1}{0}} + \underset{6}{\widetilde{T}}{}^{is}_{,s} + \frac{1}{2} \underset{2}{h}_{00;i} \underset{4}{\widetilde{T}}{}^{00} + \frac{1}{2} (\underset{4}{h}_{00,i} + \underset{2}{h}_{00} \underset{2}{h}_{00;i} - 2 \underset{3}{h}_{0i,\underset{1}{0}}) \underset{2}{\widetilde{T}}{}^{00} \\ & + (\underset{3}{h}_{0s,i} - \underset{3}{h}_{0i,s}) \underset{3}{\widetilde{T}}{}^{0s} - \underset{2}{h}_{00,\underset{1}{0}} \underset{3}{\widetilde{T}}{}^{0i} - \underset{2}{h}_{00,s} \underset{4}{\widetilde{T}}{}^{si} + \frac{1}{2} \underset{2}{h}_{00,i} \underset{4}{\widetilde{T}}{}^{ss} \Big] dV = 0. \end{aligned} \tag{5.4}$$

If into this we substitute (4.1), (4.4), and (5.3) without the rotational terms, we obtain the well-known non-Newtonian equations of translation given, for example, by Papapetrou[8] and reducing to the expressions given by Einstein[3,5] and Petrova[7] for the case of two particles. A detailed derivation, using delta-functions, is given for the two-body problem by Infeld.[10] Since the n-body problem introduces nothing fundamentally new, we limit ourselves here to a derivation of the additional rotational terms D_a^i in the equations of motion which do not appear in the detailed calculations mentioned above. If into (5.4) we substitue $\underset{2}{h}_{00}$, $\underset{3}{h}_{0i}$, and $\underset{4}{h}_{00}$, as well as the value of $\widetilde{T}^{\alpha\beta}$ to the required approximation, and take out the terms containing σ, we note that some of the integrals vanish identically while others vanish because the integrand is an odd function, and we finally obtain the simple and relatively short expression

$$D_a^i = \sum_{b\neq a} \frac{\gamma m_a m_b}{c^2}\Big\{(\dot a^s - \dot b^s)\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^i - 2(\dot a^s - \dot b^s)\Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^i + (2\dot a^s - \dot b^s)\Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s$$

$$+ (2\dot b^s - \dot a^s)\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s - [\sigma_a \times \nabla]^s\Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s + \frac{1}{4}[\sigma_a \times \nabla]^s\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s + \frac{1}{4}[\sigma_b \times \nabla]^s\Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s\Big\} \quad (5.5)$$

(here $\nabla \equiv e_i\partial/\partial a^i$). The relativistic equations of translation can now be written in the form:

$$m_a \ddot a^i = \sum_{b\neq a}\Big(\frac{\gamma m_a m_b}{r_{ab}}\Big)_{,a^i} + F_a^i + D_a^i, \quad (5.6)$$

where F_a^i are the terms to be added to the Newtonian forces if rotational effects are ignored, and which are given for instance, by Papapetrou.[8] Expressions (5.5) and (5.6) are written in the usual c.g.s. system of units. To transform to this system from the original system of units, we first eliminate the parameter λ by introducing new units of time and mass according to the formulae: old τ = new $\tau \times \lambda$; old mass = new mass $\times \lambda^{-2}$. We then introduce the usual values of Newtonian constant γ and the speed of light c. To do this we must replace m by γm, insert the factor c^{-2} on the right hand side in D_a^i, and remember that when taking time derivatives (represented by dots over the variables) the vector components are to be differentiated with respect to the usual time variable t. Then m_a and m_b are the Newtonian masses, and σ_a and σ_b are the Newtonian angular momenta of the objects (if the preceding terms $\underset{2}{m_a}$, $\underset{2}{m_b}$, $\underset{1}{\sigma_a}$, and $\underset{1}{\sigma_b}$ are constants). As to the structure of the D_a^i, notice that one group of terms (the first four terms) describes a "spin-orbit" interaction between the particles, while another group (the remaining terms) describes a "spin-spin" interaction. These interactions lead to some additional forces which will effect the orbital motions of the particles.

6. We now can derive the sixth-order equations of rotational motion. From (3.6) we have

$$\int_{V_a}[x^i(\underset{6}{\tilde T^{k\alpha}_{;\alpha}}) - x^k(\underset{6}{\tilde T^{i\alpha}_{;\alpha}})]dV = \int_{V_a}[(x^i - a^i)(\underset{6}{\tilde T^{k\alpha}_{;\alpha}}) - (x^k - a^k)(\underset{6}{\tilde T^{i\alpha}_{;\alpha}})]\,dV = 0 \quad (6.1)$$

The expressions for $\underset{6}{\tilde T^{i\alpha}_{;\alpha}}$ are given in (5.4). Relatively simple and straightforward calculations show that some of the integrals vanish,* and that the remainer lead to the following non-Newtonian equations of rotational motion (again we have transformed to the c.g.s. system):

$$\dot M_a^{ik} = \frac{\gamma}{c^2}\sum_{b\neq a} m_b\Big\{2M^{ik}_{a(0)}(\dot b^s - \dot a^s)\Big(\frac{1}{r_{ab}}\Big)_{,a^s} + M^{is}_{a(0)}\Big[(\dot a^s - 2\dot b^s)\Big(\frac{1}{r_{ab}}\Big)_{,a^k} - (\dot a^k - 2\dot b^k)\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]$$

$$- M^{ks}_{a(0)}\Big[(\dot a^s - 2\dot b^s)\Big(\frac{1}{r_{ab}}\Big)_{,a^i} - (\dot a^i - 2\dot b^i)\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big] + M^{is}_{a(0)}\Big(\frac{1}{2}\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^k - \Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^k - \frac{1}{2}\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^k}\Big]^s$$

$$+ \Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^k}\Big]^s\Big) - M^{ks}_{a(0)}\Big(\frac{1}{2}\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^i - \Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^s}\Big]^i - \frac{1}{2}\Big[\sigma_a \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s + \Big[\sigma_b \times \nabla\Big(\frac{1}{r_{ab}}\Big)_{,a^i}\Big]^s\Big)\Big\}. \quad (6.2)$$

Here, as in (5.5), m_a, m_b, ... and σ_a, σ_b, ... represent the Newtonian (i.e., constant) values of mass and specific angular momentum of the objects a, b, ...; $M^{ik}_{a(0)}$ denotes the Newtonian (constant) angular momentum of the a-th object, obtained by expressing (4.9) in the c.g.s. system. The left-hand side of (6.2) is $\dot M_a^{ik} \equiv dM_a^{ik}/dt$, the time rate of change of angular momentum of the a-th object in the first non-Newtonian approximation (including the effect of mass changes due to velocity). An expression for the angular momentum can be obtained from

$$M_a^{ik} = \lambda^3 \underset{3}{M_a^{ik}} + \lambda^5 \underset{5}{M_a^{ik}} \equiv \lambda^3\delta_{ikl}\underset{2}{m_a}\underset{1}{\sigma_a^l} + \lambda^5\delta_{ikl}(\underset{4}{m_a}\underset{1}{\sigma_a^l} + \underset{2}{m_a}\underset{3}{\sigma_a^l}) \quad (6.3)$$

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*In particular, it is found that

$$\int_{V_a}[(x^i - a^i)\underset{4}{h_{00,k}} - (x^k - a^k)\underset{4}{h_{00,i}}]\underset{2}{\tilde T^{00}}dV = 0,$$

where $\underset{4}{h_{00}}$ does not refer to the approximation of (5.3), but to its exact value. Hence we may conclude generally that $\underset{4}{h_{00}}$ does not affect the sixth-order equations of rotation.

by transforming to c.g.s. units.

The relativistic equations of rotational motion (6.2), thus developed, contain two fundamentally distinct groups of terms. One group (the first three terms) describes the "spin-orbit" interaction, while all the remaining terms describe the "spin-spin" interaction of the objects. We shall see that the existence of these interactions alters the angular momentum of spherically symmetrical objects even in the first non-Newtonian approximation.

7. It is not the object of this paper to make a detailed study of the equations of motion which have just been derived. Nevertheless, we must satisfy ourselves that they do not conflict with any known facts. For example, it is well known (cf. Landau and Lifshitz,[17] pp. 291 and 341) that in the gravitational field of an object b of mass m_b and angular momentum M_b, a light non-rotating particle a of mass m_a is acted on by a force, analogous to a Coriolis force, equal to

$$m_a\,[\dot{\mathbf{a}}\times[\nabla\times\mathbf{h}_b]],\ \text{where}\ \mathbf{h}_b=\frac{-2\gamma m_b}{c^2}\left[\boldsymbol{\sigma}_b\times\nabla\frac{1}{r_{ab}}\right]. \tag{7.1}$$

To compare (5.5) with (7.1) we have to put $\dot{b}^s = 0$, $\sigma_a = 0$, and omit the summation sign. A few transformations applied to the resulting expression does in fact lead to (7.1), plus the last term of (5.5), which has not previously been derived.

It is also interesting to compare our results, in a general way, with those obtained by Fock.[9] If Fock's extra translational and rotational terms are applied to infinitesimal spherical particles, we obtain a group of terms corresponding to the first four terms of (5.5). (We naturally ignore those terms in Fock's equations which arise from the internal structure of the objects, since this has no meaning in the present case.) Fock's equations contain no terms quadratic in σ and none of the fourth order in $1/r_{ab}$, since his method uses not only a development in the parameter v/c but also a development in the parameter L/R (L being a characteristic linear dimension of the objects and R the distance between them) and the development is broken off at terms of the third order in L/R. For large separations (as in systems of astronomical bodies) the terms in r_{ab}^{-4} are not essential; but for systems such as double stars (close, massive pairs) these terms could possibly make a significant contribution.

In conclusion, the author would like to express sincere thanks to I. Z. Fisher for proposing this subject and for his constant interest in the present work.

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POLARIZATION OF A BEAM OF MOLECULES IN AN ALTERNATING FIELD WITH VARIABLE AMPLITUDE AND PHASE

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Equations are derived which describe the behavior of a beam of molecules under the action of an alternating field with variable amplitude and phase; these equations are solved for various cases. An exact solution is obtained for the case in which the frequency of the applied field, the amplitude of which is varying, is the same as the frequency of the molecular transition. Approximate solutions are obtained for the cases in which the amplitude and frequency of the field vary slowly or rapidly during the time of flight through the cavity.

THE operation of the molecular oscillator is based on the interaction between molecules which move through a cavity and the electric field in the cavity.[1] As a result of this interaction the molecules make a transition from an upper energy state to a lower energy state, thereby giving energy to the cavity. The state of the molecular beam which interacts with the electric field in the cavity can be described by means of a polarization function. In general the relations between this polarization and the electric field in the cavity is extremely complicated. At the present time this relation has been determined only for the case in which the beam of molecules interacts with an alternating electric field whose frequency and amplitude remain constant in time. However, in analyzing all possible transient processes in a molecular oscillator and in analyzing the stability of the steady-state oscillations, it is necessary to determine the polarization of a beam which is subject to an alternating field of variable amplitude and phase (frequency).

The present paper is devoted to a theoretical study of the polarization of the beam of molecules under these conditions.

1. FORMULATION OF THE PROBLEM AND DERIVATION OF THE BASIC EQUATION

The state of a monochromatic beam of molecules, acted on by an electric field $E(t)$ is determined completely by the molecular density matrix $C_{ik}(x, t)$. The dependence of the matrix elements on the coordinate x and the time t is determined by the following system of partial differential equations:[2]

$$\frac{\partial D}{\partial t} + v\frac{\partial D}{\partial x} = -\frac{2p}{ih}[e^{i\omega_0 t}C_{12} - e^{-i\omega_0 t}C_{21}]E(t), \quad \frac{\partial C_{12}}{\partial t} + v\frac{\partial C_{12}}{\partial x} = -\frac{p}{ih}e^{-i\omega_0 t}DE(t), \quad \frac{\partial C_{21}}{\partial t} + v\frac{\partial C_{21}}{\partial x} = \frac{p}{ih}e^{i\omega_0 t}DE(t). \tag{1.1}$$

Here $D(x, t)$ is a function which characterizes the molecular distribution over energy states: $D = C_{22} - C_{11}$, v is the velocity of the molecules, p is the dipole moment of the molecules, and ω_0 is the frequency of the molecular transition.

The polarization of the beam is related to the elements of the density matrix by the expression

$$P(x, t) = p[e^{i\omega_0 t}C_{12} + e^{-i\omega_0 t}C_{21}]. \tag{1.2}$$

As is apparent from Eq. (1.2), the polarization of the beam is intimately related with the molecular transitions from one energy state to another due to the action of the cavity field. At the entrance to the cavity, where the molecules have not as yet made any energy transitions,

$$C_{12} = C_{21} = 0 \text{ for } x = 0. \tag{1.3}$$

We shall assume, moreover, that the number of active molecules which enter the resonator cavity is such that

$$D = 1 \text{ for } x = 0. \tag{1.4}$$

The further development is limited to the case in which the variation of the electric field in time is more or less sinusoidal, so that

$$E(t) = E_0(t) \cos[\omega_0 t + \varphi(t)], \tag{1.5}$$

where the field amplitude $E_0(t)$ and phase $\varphi(t)$ are slowly-varying functions of time as compared with the period $T = 2\pi/\omega_0$. Under these conditions the polarization of the beam can be written in the form of a sum of real and reactive parts:

$$P(x, t) = P_1(x, t) \sin[\omega_0 t + \varphi] + P_2(x, t) \cos[\omega_0 t + \varphi]. \tag{1.6}$$

The amplitudes of the real and reactive parts, P_1 and P_2, are slowly-varying functions of time. If, in the right halves of Eqs. (1.1) we neglect terms which vary rapidly in time (frequency $2\omega_0$) and introduce new variables defined by the relations

$$\xi = x - vt; \quad \eta = x + vt, \tag{1.7}$$

the system of equations in (1.1) assumes the form

$$\frac{\partial D}{\partial \eta} = \frac{E_0 p}{2ihv}[e^{i\varphi}C_{21} - e^{-i\varphi}C_{12}], \quad \frac{\partial C_{12}}{\partial \eta} = -\frac{p}{4ihv} e^{i\varphi} E_0 D, \quad \frac{\partial C_{21}}{\partial \eta} = \frac{p}{4ihv} e^{-i\varphi} E_0 D. \tag{1.8}$$

In this case the boundary conditions given in (1.3) and (1.4) assume the form

$$D = 1, \; C_{12} = C_{21} = 0 \text{ for } \eta = -\xi. \tag{1.9}$$

The values of the elements of the density matrix C_{12} and C_{21} and consequently the polarization of the beam $P(x, t)$, in accordance with the second and third equations in (1.8), can be expressed in terms of the function D as follows:

$$C_{12} = -\frac{p}{4ihv} \int_{-\xi}^{\eta} e^{i\varphi(\zeta)} E_0(\zeta) D(\zeta)\, d\zeta, \quad C_{21} = \frac{p}{4ihv} \int_{-\xi}^{\eta} e^{-i\varphi(\zeta)} E_0(\zeta) D(\zeta)\, d\zeta. \tag{1.10}$$

Substituting (1.10) in (1.2) and, in accordance with (1.6), dividing the polarization into real and reactive parts, we obtain the expressions

$$P_1 = -\frac{p^2}{2hv} \int_{-\xi}^{\eta} \cos[\varphi(\zeta) - \varphi(\eta)]\, E_0(\zeta) D(\zeta) d\zeta, \quad P_2 = -\frac{p^2}{2hv} \int_{-\xi}^{\eta} \sin[\varphi(\zeta) - \varphi(\eta)]\, E_0(\zeta) D(\zeta)\, d\zeta. \tag{1.11}$$

Whence it is easy to show that the system given in (1.8) is equivalent to the following system of equations:

$$\frac{\partial D}{\partial \eta} = \frac{1}{2hv} E_0(\eta) P_1, \quad \frac{\partial P_1}{\partial \eta} = \varphi'(\eta) P_2 - \frac{p^2}{2hv} E_0(\eta) D, \quad \frac{\partial P_2}{\partial \eta} = -1\, \varphi'(\eta) P_1 \tag{1.12}$$

with the boundary conditions at $x = 0$

$$D = D_0, \quad P_1 = P_2 = 0. \tag{1.13}$$

The function D is described by an integro-differential equation which can be obtained by substituting (1.10) in the first equation of (1.8):

$$\frac{\partial D}{\partial \eta} = -\gamma^2 E_0(\eta) \int_{-\xi}^{\eta} E_0(\zeta) \cos[\varphi(\eta) - \varphi(\zeta)]\, D(\zeta)\, d\zeta; \tag{1.14}$$

here $\gamma = p/2hv$.

On the other hand, the function D obeys the differential equation

$$\frac{d^2}{d\eta^2}\left[\frac{1}{E_0(\eta)} \frac{\partial D}{\partial \eta}\right] - \frac{\varphi''(\eta)}{\varphi'(\eta)} \frac{d}{d\eta}\left[\frac{1}{E_0(\eta)} \frac{\partial D}{\partial \eta}\right] + \gamma^2 \frac{d}{d\eta}[E_0(\eta) D(\eta)] + \frac{\varphi'^2(\eta)}{E_0(\eta)} \frac{\partial D}{\partial \eta} - \gamma^2 \frac{\varphi''(\eta)}{\varphi'(\eta)} E_0(\eta) D(\eta) = 0 \tag{1.15}$$

with the boundary conditions

$$D = 1, \quad D' = 0, \quad D'' = -\gamma^2 E_0^2(-\xi) \quad \text{for } \eta = -\xi, \tag{1.16}$$

which can be obtained starting from the system in (1.12) for the functions P_1 and P_2. The equivalence of (1.14) and (1.15) is easily demonstrated. Thus, the determination of the polarization of the beam of molecules being acted on by an electric field such as that given in (1.5) leads to the solution of the differential equation in (1.15). This equation is not amenable to solution in the general case in which both the amplitude and phase of the field vary in time. However, Eq. (1.15) can be solved if certain limitations are imposed on the amplitude and phase of the field. The remainder of this paper is devoted to such solutions.

2. APPLIED FIELD WITH CONSTANT PHASE

In the case in which the frequency of the electric field acting on the beam of molecules is equal to the frequency of the molecular transition but the amplitude E_0 varies in some arbitrary manner the solution of Eq. (1.15) can be found by quadratures. In Eq. (1.15) we set $\varphi = \text{const} = 0$. Then the equation assumes the form

$$\frac{d^2}{d\eta^2}\left[\frac{1}{E_0(\eta)}\frac{\partial D}{\partial \eta}\right] + \gamma^2 \frac{d}{d\eta}\left[E_0(\eta)\, D(\eta)\right] = 0\,. \tag{2.1}$$

Integrating Eq. (2.1) once and setting the constant of integration equal to zero, we have

$$\frac{d}{d\eta}\left[\frac{1}{E_0(\eta)}\frac{\partial D}{\partial \eta}\right] + \gamma^2 E_0(\eta)\, D(\eta) = 0\,. \tag{2.2}$$

As can be shown by direct substitution, the general solution for this equation is given by the expression

$$D = A\cos\left[\gamma \int\limits_{-\xi}^{\eta} E_0(\zeta)\, d\zeta\right] + B\sin\left[\gamma \int\limits_{-\xi}^{\eta} E_0(\zeta)\, d\zeta\right]. \tag{2.3}$$

Here A and B are constants of integration. The boundary conditions in (1.8) are satisfied if A = 1 and B = 0.

In accordance with (1.12), in the constant-phase case the reactive part of the polarization P_2 vanishes while the real part becomes

$$P_1 = 2hv\,\frac{1}{E_0}\,\frac{\partial D}{\partial \eta} = -\,p\sin\left[\gamma \int\limits_{-\xi}^{\eta} E_0(\zeta)\, d\zeta\right]. \tag{2.4}$$

If we now transform back to the variables x and t, the argument of the sine function in (2.4) assumes the form

$$2v\gamma \int\limits_{0}^{x/v} E_0(t - t')\, dt'. \tag{2.5}$$

An examination of this expression shows that the time variations of the field amplitude are averaged over an internal x/v which is of the order of magnitude of the molecular time of flight across the cavity. This situation is the basis for the "inertia non-linearity" of a molecular oscillator where the time constant of the "inertia" is equal to the time of flight across the cavity. It also follows from Eq. (2.5) that the quasi-static expression for the function D

$$D = \cos 2\gamma\, Ex,$$

as well as the corresponding expression for the polarization of the beam

$$P(x, t) = -\,p\sin \omega_0 t \sin 2\gamma\, Ex,$$

can be used in the case in which the amplitude of the field varies very slowly,

$$\tau E' / E \ll 1.$$

It will be shown below that these results can be extended to the general case in which both the amplitude and phase of the field vary although Eq. (1.15) cannot then be solved by quadratures.

3. APPLIED FIELD WITH SLOWLY VARYING AMPLITUDE AND PHASE

In the case in which the amplitude E_0 and frequency φ' vary slowly as compared with the time interval τ, it is possible to find an asymptotic solution for Eq. (1.15); the method is similar to that used to find the quasi-classical approximations for the Ψ-function in quantum mechanics[3] or the geometric-optic methods used in the analysis of the propagation of electromagnetic waves.[4] For this purpose we write the solution of Eq. (1.15) in the form

$$D(\eta) = e^{i\Phi(\eta)}. \tag{3.1}$$

The equation which describes the function $\Phi(\eta)$ is of the form:

$$-i\Phi'^3 - 3\Phi'\Phi'' + i\Phi''' - \left[2\frac{E'}{E} + \frac{\varphi''}{\varphi'}\right][i\Phi'' - \Phi'^2] + \left[\gamma^2E^2 + \varphi'^2 + 2\left(\frac{E'}{E}\right)^2 - \frac{E''}{E} + \frac{\varphi''}{\varphi'}\right]i\Phi' + \gamma^2E^2\left[\frac{E'}{E} - \frac{\varphi''}{\varphi'}\right] = 0. \tag{3.2}$$

In the case in which E and φ' are constant in time, the function D is a periodic function of η, with frequency[2]

$$\Omega = \sqrt{\gamma^2E^2 + \varphi'^2}. \tag{3.3}$$

If the amplitude and phase of the field do not vary significantly over a period of the function D, we can introduce a large parameter λ, which characterizes the rate of change of the function D as compared with the variations of the functions E and φ':

$$\gamma E \sim \lambda, \quad \varphi' \sim \lambda, \quad E'/E \sim 1, \quad \varphi''/\varphi' \sim 1. \tag{3.4}$$

In this case, the function Φ in Eq. (3.1) can be expanded in powers of λ

$$\Phi = \lambda\Phi_0(\eta) + \Phi_1(\eta) + \frac{1}{\lambda}\Phi_2(\eta) + \ldots. \tag{3.5}$$

Substituting (3.4) in Eq. (3.1), expanding the left half in powers of λ, taking account of (3.4), and equating terms with the same power of λ, we have

$$\lambda^3) \qquad -\Phi_0'^3 + (\gamma^2E^2 + \varphi'^2)\Phi_0' = 0, \tag{3.6}$$

$$\lambda^2) - 3\Phi_0'\Phi_0'' - 3i\Phi_0'^2\Phi_1' + 2\frac{E'}{E}\Phi_0'^2 + \frac{\varphi''}{\varphi'}\Phi_0'^2 + i\gamma^2E^2\Phi_1' + i\varphi'^2\Phi_1' - \gamma^2E^2\frac{\varphi''}{\varphi'} + \gamma^2E^2\frac{E'}{E} = 0. \tag{3.7}$$

The solutions of these equations are of the form:

$$\Phi_{01}' = 0; \quad \Phi_{02.3}' = \pm\sqrt{\gamma^2E^2 + \varphi'^2}, \quad \Phi_{11} = -i\ln\frac{\varphi'}{\sqrt{\gamma^2E^2 + \varphi'^2}}; \quad \Phi_{12.3} = -i\ln\frac{\gamma E}{\sqrt{\gamma^2E^2 + \varphi'^2}}. \tag{3.8}$$

Substituting these solutions in (3.1) and taking account of the boundary conditions (1.16) we can obtain an approximate expression for the function D in the case in which the amplitude and frequency of the field which acts on the beam are slowly varying

$$D(x,t) = \frac{\gamma^2E(t)E\left(t - \frac{x}{v}\right)\cos\int\limits_{t-x/v}^{t} 2v\left(\gamma^2E^2(t_1) + \frac{\varphi'^2(t_1)}{4v^2}\right)^{1/2}dt_1 + \frac{\varphi'(t)\varphi'(t - x/v)}{4v^2}}{[\gamma^2E^2(t) - \varphi'^2(t)/4v^2]^{1/2}[\gamma^2E^2(t - x/v) - \varphi'^2(t - x/v)/4v^2]^{1/2}}. \tag{3.9}$$

This expression, which applies when

$$\tau\varphi''/\sqrt{\gamma^2E^2 + \varphi'^2} \ll 1, \quad \tau\gamma E'/\sqrt{\gamma^2E^2 + \varphi'^2} \ll 1, \tag{3.10}$$

indicates that the beam tends to average the effect of the field with variable amplitude and frequency during the time in which the molecules move through the cavity; this effect, however, is small because the changes are slow. Substituting Eq. (3.9) in Eq. (1.12) it is easy to determine the polarization of the beam.

4. APPLIED FIELD WITH RAPIDLY VARYING AMPLITUDE AND FREQUENCY

Equation (1.15) can also be solved approximately when there are very rapid periodic changes in the amplitude and frequency of the field which acts on the beam, i.e., changes fast as compared with the time of flight of the molecules through the cavity. Let

$$E_0 = E_1 + e(\lambda, t); \; \varphi' = \varphi_1' + \chi(\lambda, t), \tag{4.1}$$

where E_1 and $\varphi_1' = \delta/2v$ are constants ($E_1 \neq 0$, $\delta = \omega - \omega_0$), which characterize the mean amplitude and frequency deviation of the field while $e(t)$ and $\chi(t)$ are periodic functions of time whose periods are smaller by a factor of λ than the molecular time of flight through the cavity ($\lambda \gg 1$).

In solving Eq. (1.15) for $D(\eta)$ in this case we can separate terms which vary rapidly with a change in the argument from those which vary slowly:

$$D(\eta) = D_0(\eta) + \frac{1}{\lambda} D_1(\eta) + \ldots + \frac{1}{\lambda} d_0(\lambda, \eta) + \frac{1}{\lambda^2} d_2(\lambda, \eta) + \ldots \tag{4.2}$$

We substitute (4.1) and (4.2) in Eq. (1.14) and collect terms in the same power of the parameter λ. In the zeroth approximation we have

$$D_0' + d_0' = -\gamma^2 E_1^2 \int_{-\xi}^{\eta} \cos\delta(\eta - \zeta)\, D_0(\zeta)\, d\zeta - \gamma^2 E_1 e(\lambda, \eta) \int_{-\xi}^{\eta} \cos\delta(\eta - \zeta) D_0(\zeta) d\zeta. \tag{4.3}$$

Separating rapidly-varying and slowly-varying terms in this expression we obtain

$$D_0' = \gamma^2 E_1^2 \int_{-\xi}^{\eta} \cos\delta(\eta - \zeta)\, D_0(\zeta)\, d\zeta, \tag{4.4}$$

$$d_0' = (e/E_1)\, D_0' \cdot \tag{4.5}$$

The solution of Eq. (4.4) indicates that in the zeroth approximation rapid variations in the amplitude and frequency of the field which acts on the beam have no effect on the distribution of the molecules over states—the field is averaged. The rapid time variations in the molecular distribution over states are small in absolute magnitude and are determined, in accordance with (4.5), by the change in the field amplitude. The rapid changes in the frequency of the field affect the function $D(\eta)$ only to second order in $1/\lambda$.

In a similar fashion we can extend the calculations to the higher approximations. By this means it can be shown that the slowly-varying part $D_1(\eta) = 0$.

Hence, with accuracy up to terms of order $1/\lambda$ inclusive, for a field which is characterized by rapid changes in amplitude and frequency the solution of Eq. (1.15) is given by the relation:

$$D(\eta) = D_0(\eta) + \frac{1}{\lambda} d_0(\lambda, \eta). \tag{4.6}$$

The polarization of the beam, given by the expression

$$P = -p\gamma \int_{-\xi}^{\eta} \sin[\omega_0 t + \varphi(\zeta)]\, E_1(\zeta) D(\zeta)\, d\zeta, \tag{4.7}$$

can now be found easily. With accuracy to first order this quantity is

$$P = P_s + P_f, \tag{4.8}$$

where P_s is the part of the polarization which varies slowly with time

$$P_s = -p\gamma \frac{1}{\Omega^2} \left\{ \Omega \sin 2\Omega x \,[E_1 \sin\omega t + \overline{e\chi} \cos\omega t] + \delta(1 - \cos 2\Omega x)\,[E_1 \cos\omega t + \overline{e\chi} \sin\omega t] \right\}. \tag{4.9}$$

Here $\overline{e\chi}$ is the mean value of the product of the amplitude deviation and the frequency deviation.

The rapidly varying part of the polarization is given by the expression:

$$P_f = -\frac{p^2}{h} D_0(x) \int_{t - x/v}^{t} e(t')\, dt'. \tag{4.10}$$

In conclusion we may note that the relations which have been obtained for the polarization of a beam of molecules are rather simple and can be used directly in studying quasi-periodic modes in the molecular oscillator; an investigation of this kind, however, is beyond the scope of the present paper.

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CONTRIBUTION TO THE THEORY OF TRANSITION RADIATION

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The transition radiation and the Cerenkov radiation which are produced when a charged particle moves successively through two media which differ in their dielectric and magnetic properties are considered. The cases in which the particle moves from vacuum into the medium and from the medium into vacuum are considered in detail.

THE transition radiation which is produced when a particle moves from a medium characterized by a given dielectric constant into another whose dielectric constant differs from the first was first considered by Ginzburg and Frank[1] (see also Refs. 2–4). In the present paper we consider the radiation fields which are produced in the general case for media which differ in both their dielectric and magnetic properties; certain particular cases are analyzed.

1. GENERAL CASE

We consider the field associated with a particle which has a velocity **v** and moves from one medium into another. The first medium will be characterized by the macroscopic constants ϵ_1 and μ_1 (the dielectric constant and magnetic permeability); the second medium is characterized by ϵ_2 and μ_2. We shall assume that the energy lost by the particle per unit length of path is negligibly small compared with its kinetic energy. Under these conditions the field associated with the particle is given by Maxwell's equations

$$\begin{aligned} \operatorname{curl}\mathbf{H} &= \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{v}\,e\delta(\mathbf{r}-\mathbf{v}t), & \operatorname{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \\ \operatorname{div}\mathbf{B} &= 0 & \operatorname{div}\mathbf{D} &= 4\pi e\delta(\mathbf{r}-\mathbf{v}t). \end{aligned} \tag{1}$$

It will be assumed that the particle moves along the z-axis from $-\infty$ to $+\infty$ and that the interface between the two media is the plane $z = 0$ through which the particle moves at $t = 0$. We resolve the field and currents in triple Fourier integrals:[5]

$$\mathbf{E}(\mathbf{r}, t) = \int \mathbf{E}(\mathbf{k}) e^{i(\mathbf{kr} - \omega t)} d\mathbf{k} \quad \text{etc.,} \tag{2}$$

where

$$\omega = \mathbf{kv} = k_z v, \quad \mathbf{D}_{1,2}(\mathbf{k}) = \varepsilon_{1,2}(\omega) \mathbf{E}_{1,2}(\mathbf{k}), \quad \mathbf{B}_{1,2}(\mathbf{k}) = \mu_{1,2}(\omega) \mathbf{H}_{1,2}(\mathbf{k})$$

(here and in what follows, quantities which refer to both media will be denoted by the subscripts 1, 2).

The Fourier components of the field have the following form:[6]

$$\mathbf{E}_{1,2}(\mathbf{k}) = \frac{ei}{2\pi^2} \frac{1}{\varepsilon_{1,2}} \frac{(\omega / c^2) \chi_{1,2} \mathbf{v} - \mathbf{k}}{k^2 - (\omega / c)^2 \chi_{1,2}}, \quad \mathbf{H}_{1,2}(\mathbf{k}) = (\varepsilon_{1,2} / c) [\mathbf{v} \times \mathbf{E}_{1,2}(\mathbf{k})], \tag{3}$$

where $\chi_{1,2} = \epsilon_{1,2}\mu_{1,2}$. The fields in (2) with Fourier components (3) do not satisfy the continuity conditions on the tangential components of E and H and on the normal components of D and B at $z = 0$. To satisfy these requirements we must add to the solution of the inhomogeneous Maxwell equations given above the solutions of the homogeneous equations with arbitrary Fourier component, and then determine these from the continuity requirements on the fields at the interface between the two media.

We use the symbols ρ and κ to denote the components of the vectors **r** and **k** which lie in the xy-plane. The solutions of the homogeneous Maxwell equations are written in the form

$$\mathbf{E}'_{1,2}(\mathbf{r}, t) = \int \mathbf{E}'_{1,2}(\mathbf{k}) \exp\{i(\varkappa\rho + \lambda_{1,2} z - \omega t)\} d\mathbf{k} \tag{4}$$

and similarly for $H'_{1,2}(r, t)$. In order for the expression given in (4) to be a solution of the homogeneous Maxwell equations, we require that

$$\lambda_{1,2}^2 = (\omega / c)^2 \chi_{1,2} - \varkappa^2. \tag{5}$$

We use the symbol λ' to denote the real part of λ and λ'' to denote the imaginary part of λ. The first medium is located in the region $z < 0$. Hence, to prevent the field given in (4) from diverging at $z \to -\infty$, we require that $\lambda_1'' < 0$. It is also obvious that the radiation field in the first medium (4) can propagate only in the negative z-direction (reflected waves) whence it follows that $\lambda_1' < 0$. From similar considerations we find that $\lambda_2' > 0$ and $\lambda_2'' > 0$. The signs for λ_1' and λ_2' which have been indicated refer to positive ω. For negative values of ω these signs must be reversed.

It also follows from the equations for the radiation fields that

$$\mathbf{H}'_{1,2}(\mathbf{k}) = (c / \omega\mu_{1,2})[(\varkappa + \mathbf{n}\lambda_{1,2}) \times E'_{1,2}(\mathbf{k})], \quad (\varkappa + \mathbf{n}\lambda_{1,2}) \mathbf{E}'_{1,2}(\mathbf{k}) = 0 \tag{6}$$

(the unit vector **n** is taken in the direction of the positive z-axis). The last condition can be written in another form, resolving $E'_{1,2}(k)$ into tangential and normal components:

$$\varkappa \mathbf{E}'_{1,2t}(\mathbf{k}) + \lambda_{1,2} E'_{1,2n}(\mathbf{k}) = 0. \tag{7}$$

Equating the field components at $z = 0$, we obtain four conditions for determining the Fourier components of the radiation field. It is easy to show from these conditions that the $E'_{1,2t}(k)$ vectors are in the same direction as the vector κ. Assuming this to be the case, it turns out that only two of the four conditions are independent; we take the following two conditions:

$$\begin{gathered} -\frac{ei}{2\pi^2} \frac{1}{\varepsilon_1} \frac{\varkappa}{k^2 - (\omega / c)^2 \chi_1} + E'_{1t}(\mathbf{k}) = -\frac{ei}{2\pi^2} \frac{\varkappa}{k^2 - (\omega / c)^2 \chi_2} + E'_{2t}(\mathbf{k}), \\ \frac{ei}{2\pi^2} \frac{(\omega / c^2) \chi_1 v - k_z}{k^2 - (\omega / c)^2 \chi_1} + \varepsilon_1 E'_{1n}(\mathbf{k}) = \frac{ei}{2\pi^2} \frac{(\omega / c^2) \chi_2 v - k_z}{k^2 - (\omega / c)^2 \chi_2} + \varepsilon_2 E'_{2n}(\mathbf{k}). \end{gathered} \tag{8}$$

From Eq. (8) we obtain the following expressions for the Fourier components of the radiation field:

$$\mathbf{E}'_{1t}(\mathbf{k}) = \frac{ei}{2\pi^2} \frac{\varkappa\lambda_1}{\zeta} \eta, \quad E'_{1n} = -\frac{ei}{2\pi^2} \frac{\varkappa^2}{\zeta} \eta, \quad \mathbf{H}_1(\mathbf{k}) = -\frac{ei}{2\pi^2 c} \frac{k_z \varepsilon_1 [\varkappa \mathbf{v}]}{\zeta} \eta. \tag{9}$$

The following notation has been introduced

$$\eta = \left(\frac{\varepsilon_2}{\varepsilon_1} - \frac{v}{\omega} \lambda_2\right) \Big/ \left(k^2 - \frac{\omega^2}{c^2} \chi_1\right) + \left(-1 + \frac{v}{\omega} \lambda_2\right) \Big/ \left(k^2 - \frac{\omega^2}{c^2} \chi_2\right), \quad \zeta = \varepsilon_2 \lambda_1 - \varepsilon_1 \lambda_2. \tag{10}$$

The radiation fields in the second medium can be obtained from Eqs. (9) and (10) if the subscripts 1 and 2 are interchanged. If in Eq. (9) we take $\mu_1 = \mu_2$ and $\epsilon_1 = \epsilon_2$, all the Fourier components of the radiation field vanish, as is to be expected.

2. VACUUM-TO-MEDIUM CASE

We consider now a case which is of practical interest, i.e., the case in which the particle moves from vacuum into a medium; we set $\epsilon_1 = \mu_1 = 1$, $\mu_2 = 1$ and $\epsilon_2 = \epsilon = \epsilon' + \epsilon''$. We shall be interested in the field which is produced in the vacuum, i.e., the radiation field E_1' and H_1'.

We shall not write expressions for all the vacuum fields, as these can be easily obtained from Eqs. (9) and (10); the expression for E_1' is:

$$E_{1\rho}' = \frac{ei}{2\pi^2}\int \frac{\varkappa\lambda_1 \cos\Phi}{\varepsilon\lambda_1 - \lambda_2}\eta_1 \exp\{i(\varkappa\rho\cos\Phi + \lambda_1 z - \omega t)\}\, \varkappa\, d\varkappa\, d\Phi\, \frac{d\omega}{v}; \tag{11}$$

where Φ is the angle between κ and ρ, while

$$\eta_1 = \left(\varepsilon - \frac{v}{\omega}\lambda_2\right) \Big/ \left(\varkappa^2 + \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\right) + \left(-1 + \frac{v}{\omega}\lambda_2\right) \Big/ \left(\varkappa^2 + \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\varepsilon\right), \quad \lambda_1^2 = (\omega/c)^2 - \varkappa^2, \quad \lambda_2^2 = (\omega/c)^2\varepsilon - \varkappa^2. \tag{12}$$

The integration over Φ extends from 0 to 2π, that over κ from 0 to ∞, and that over ω from $-\infty$ to $+\infty$. The integral over Φ is easily computed in terms of Bessel functions.

We introduce R, the distance from the point at which the particle enters the medium to the field point being investigated, and the angle θ defined by the expression $\rho = R\sin\theta$ and $z = -R\cos\theta$; it is assumed that R is large. If very small values of θ are not considered the asymptotic expressions for the Bessel functions[7] can be used:

$$J_p(\varkappa R\sin\theta) = \sqrt{\frac{2}{\pi\varkappa R\sin\theta}}\cos\left(\varkappa R\sin\theta - \frac{p\pi}{2} - \frac{\pi}{4}\right). \tag{13}$$

Thus we have

$$E_{1\rho}' = -\frac{e}{\pi v\sqrt{2\pi R\sin\theta}}\int \frac{\varkappa\lambda_1}{\varepsilon\lambda_1 - \lambda_2}\eta_1 \left(e^{f(\varkappa)R - 3\pi i/4} + e^{\varphi(\varkappa)R + 3\pi i/4}\right) e^{-i\omega t}\sqrt{\varkappa}\, d\varkappa\, d\omega; \tag{14}$$

$$f(\varkappa) = i\varkappa\sin\theta - i\lambda_1\cos\theta; \quad \varphi(\varkappa) = -i\varkappa\sin\theta - i\lambda_1\cos\theta. \tag{15}$$

For very large values of R, an integral of this type is most conveniently computed by the method of steepest descent. In this case the integrand must be an analytic function of κ. However, the presence of λ_1 and λ_2 in the integrand mean that the integrand is double valued. Hence, before deforming the path of integration we must take cuts in the κ-plane to make this function single valued.*

Above we have imposed certain conditions on the signs of the real and imaginary parts $\lambda_{1,2}$ along the real κ axis. We have taken $\kappa = \kappa_1 + i\kappa_2$. We require that $\lambda_1' < 0$ and $\lambda_2' > 0$ over the entire plane of the complex variable κ. It is easy to show that the sign of λ_2'' must correspond to that shown in Fig. 1 so that $\lambda_1'' > 0$ in the 1-st and 3-rd quadrants and $\lambda_1'' < 0$ in the 2-nd and 4-th quadrants. The cuts are taken as shown in Fig. 1 by the heavy lines. The integration over κ from the point 2 to $+\infty$ is taken along the lower edge of the cut.

We now turn to the integral in (14), proceeding with the integration of the first part

$$e^{3\pi i/4}\int_0^\infty \frac{\varkappa\lambda_1}{\varepsilon\lambda_1 - \lambda_2}\eta_1 e^{f(\varkappa)R}\sqrt{\varkappa}\, d\varkappa. \tag{16}$$

It is easy to show that the saddle point is $\kappa_0 = (\omega/c)\sin\theta$. Two lines on which $\operatorname{Re} f(\kappa) = 0$ pass through this point. One of these lines is the κ_1-axis while the other is the solid line shown in Fig. 2. In the cross-hatched regions $\operatorname{Re} f(\kappa) > 0$; in the non-cross-hatched regions $\operatorname{Re} f(\kappa) < 0$. It can be shown that the line of shortest descent in the vicinity of the saddle point is at the angle $\varphi = -\pi/4$ with respect to the κ_1 axis, as shown by the dotted line in Fig. 2. We introduce a new variable b which is defined by the

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*The ambiguity due to the presence of $\sqrt{\kappa}$ in Eq. (16) is easily removed by taking the cut along the negative κ-axis.

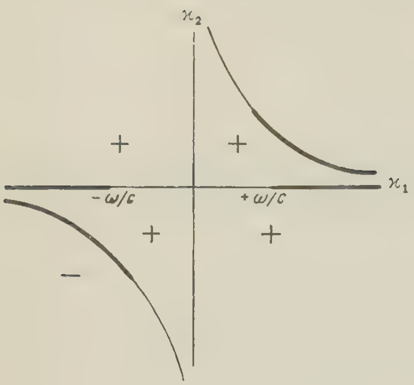

FIG. 1

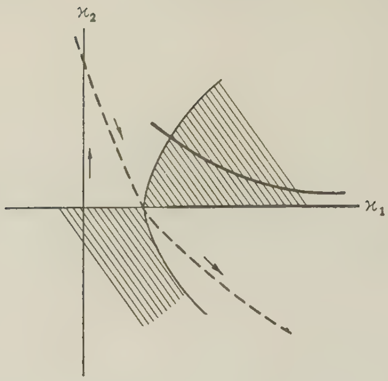

FIG. 2

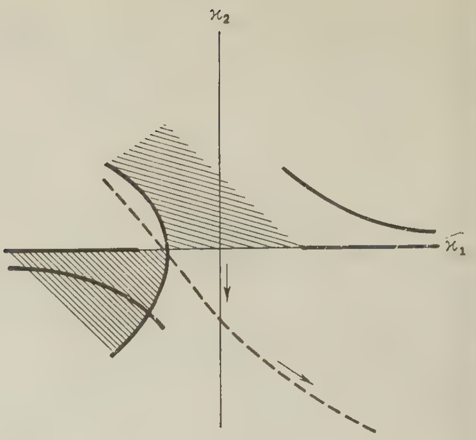

FIG. 3

relation $\kappa - \kappa_0 = b \exp i\varphi$. The integral in (16) along the line of shortest descent is easily computed if we carry out the integration over b from $-\infty$ to $+\infty$. In this case, because of the sharp maximum in the exponential at the saddle point, the expression which stands before the exponential may be taken outside the integral sign since it varies only slightly in this region and we then perform the integration for the exponential function alone.

It is still necessary to integrate the second term in Eq. (14), in the exponential of which we have the function $\varphi(\kappa)$. In a completely analogous way it can be shown that the saddle point is now $\kappa_0 = (\omega/c)\sin\varphi$, while the dotted line in Fig. 3 is the line of shortest descent. It is easy to see, however, that in this case the deformed path of integration does not pass through the saddle point (Fig. 3), so that this integral vanishes in the approximation of large R. It is also easy to show that in this case the poles of the integrand do not affect the calculations.

As a result we have

$$E'_{1\rho} = \frac{e\beta^2}{\pi v R}\int_{-\infty}^{+\infty} \sin\theta\cos^2\theta\,\xi e^{i\omega[(R/c)-t]}\,d\omega, \tag{17}$$

where

$$\xi = \frac{(\varepsilon - \beta\sqrt{\varepsilon - \sin^2\theta})/(1-\beta^2\cos^2\theta) - 1/(1+\beta\sqrt{\varepsilon-\sin^2\theta})}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}. \tag{18}$$

Similar calculations lead to the following expression for the normal component of the radiation field:

$$E'_{1n} = \frac{e\beta^2}{\pi v R}\int_{-\infty}^{+\infty} \sin^2\theta\cos\theta\,\xi e^{i\omega[(R/c)-t]}\,d\omega. \tag{19}$$

It is obvious from the formulas which have been obtained that the electric vector of the radiation field E'_1 lies in the plane which passes through the ray to the point of observation and the trajectory of the particle and is perpendicular to the observation ray, i.e., the direction of R. We have

$$E'_1 = E'_{1\rho}\cos\theta + E'_{1n}\sin\theta = \frac{e\beta^2}{\pi v R}\int_{-\infty}^{+\infty} \sin\theta\cos\theta\,\xi e^{i\omega[(R/c)-t]}\,d\omega. \tag{20}$$

Finally, we can also obtain an expression for the magnetic field H'_1 which, as is to be expected, is the same as the expression for E'_1. Thus, at large distances, we have a spherically diverging wave with Poynting vector parallel to R. The Poynting vector flux in the solid angle $d\Omega = \sin\theta\, d\theta\, d\varphi$ during the time of flight of the particle is

$$\frac{dW_{\text{trans.}}}{d\Omega} = \frac{c}{4\pi}R^2\int_{-\infty}^{+\infty} E'_1 H'_{1\varphi}\,dt = \frac{ce^2\sin^2\theta\cos^2\theta}{\pi^2 v^2}\,\frac{\beta^4}{(1-\beta^2\cos^2\theta)^2}\int_0^{\infty}\left|\frac{(\varepsilon-1)(1-\beta^2+\beta\sqrt{\varepsilon-\sin^2\theta})}{(\varepsilon\cos\theta+\sqrt{\varepsilon-\sin^2\theta})(1+\beta\sqrt{\varepsilon-\sin^2\theta})}\right|^2 d\omega. \tag{21}$$

The expression which has been obtained coincides with the corresponding formula in the paper by Ginzburg and Frank,[1] which was obtained in a more approximate manner.*

In the extreme relativistic case the radiation has a sharp maximum in the direction $\theta \sim m/E$. Thus, in integrating Eq. (21) over angle, it is convenient to remove all factors which have a weak angular dependence from under the integral sign, substituting $\theta \sim 0$ in these expressions. The result is

$$W_{\text{trans.}} = \frac{e^2}{\pi c}\int_0^\infty \left(\frac{\sqrt{\varepsilon(\omega)}-1}{\sqrt{\varepsilon(\omega)}+1}\right)^2 \left(\ln\frac{2}{1-\beta}-1\right) d\omega. \tag{22}$$

In conclusion we may note that in integrating over κ certain restrictions have been applied; the results must therefore be qualified. In particular, we have taken the asymtotic expansions in place of the Bessel function and have used method of steepest descent in the integration. It is obvious that this procedure is valid when $(\omega/c)R\sin^2\theta \gg 1$. Thus, in the spatial region close to the trajectory of the particle, bounded by the surface $R \sim \lambdabar/\sin^2\theta$, the Poynting vector associated with the transition radiation is not given by Eq. (21). In this region there occurs the "formation" of the transition-radiation wave field which then, at distances $R \gg \lambdabar/\sin^2\theta$, is given by Eq. (21). The existence of this region is unimportant for that part of the transition radiation which is not emitted at small angles. However, in the extreme relativistic case, in which the radiation has a sharp maximum in the direction $\theta \sim m/E$, this "formation" region is extremely important and is defined by $R \sim \lambdabar E^2/m^2$.

3. MEDIUM-TO-VACUUM CASE

We now consider the case in which the particle moves from medium into vacuum, i.e., $\epsilon_2 = \mu_2 = 1$, $\mu_1 = 1$ and $\epsilon_1 = \epsilon = \epsilon' + i\epsilon''$. It is obvious that in this case the radiation field in the vacuum will consist of both the field due to transition radiation as well as that due to the Cerenkov radiation which is generated in the medium and then propagates into the vacuum.†

Thus, just as in the preceding section we write an expression for $E'_{2\rho}$, integrate over angle Φ and take the values at large distances R ($\rho = R\sin\theta$, $z = R\cos\theta$):

$$E'_{2\rho} = \frac{e}{\pi v\sqrt{2\pi R\sin\theta}}\int \frac{\varkappa\lambda_2}{\varepsilon\lambda_2-\lambda_1}\eta_2\left(e^{f_1(\varkappa)R-3\pi i/4} + e^{\varphi_1(\varkappa)R+3\pi i/4}\right)e^{-i\omega t}\sqrt{\varkappa}\,d\varkappa\,d\omega, \tag{23}$$

where

$$\eta_2 = \left(1-\frac{v}{\omega}\lambda_1\right)\Big/\left(\varkappa^2+\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\varepsilon\right) + \left(-\varepsilon+\frac{v}{\omega}\lambda_1\right)\Big/\left(\varkappa^2+\frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\right), \quad \lambda_1^2 = \omega^2\varepsilon/c^2-\varkappa^2, \quad \lambda_2^2 = \omega^2/c^2-\varkappa^2,$$
$$f_1(\varkappa) = i\varkappa\sin\theta + i\lambda_2\cos\theta, \quad \varphi_1(\varkappa) = -i\varkappa\sin\theta + i\lambda_2\cos\theta. \tag{24}$$

We set $\lambda'_1 < 0$ and $\lambda'_2 > 0$ over the entire plane of the complex variable κ. Then the signs of λ''_1 and λ''_2 are the opposite of λ''_2 and λ''_1 respectively, as given in the preceding section.

It is easy to show that the saddle points and lines of steepest descent of the functions $f_1(\kappa)$ and $\varphi_1(\kappa)$ coincide with the saddle points and lines of steepest descent for the functions $f(\kappa)$ and $\varphi(\kappa)$. However, in contrast to the preceding case, the following two factors must be kept in mind in the integration over κ.

First, in deforming the line $0-\infty$ in the line of steepest descent, account must be taken in certain cases of the pole of the function η_2, which adds the residue at this pole to the integral along the saddle-point line. Such a pole of a function η_2 is $\kappa' = (\omega/v)\sqrt{\beta^2\epsilon(\omega)-1}$. Since $\epsilon'' \geq 0$, the pole contributes to the integral in this case if it is located in the κ plane in the region which is cross-hatched in Fig. 4. It will be shown in the following that this term yields the Cerenkov radiation which is generated in the medium and then passes into the vacuum. For small values of ϵ''

$$\varkappa' = \frac{\omega}{v}\left[\sqrt{\beta^2\varepsilon'-1} + i\beta^2\varepsilon''/\sqrt{\beta^2\varepsilon'-1}\right]. \tag{25}$$

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*The above-mentioned formula [(32) of Ref. 1] contains an error: the denominator of the first term in circular brackets should read $\omega/v + \sqrt{k_2^2 - k_1^2\sin^2\theta}$ and not $\omega/v - \sqrt{k_2^2 - k_1^2\sin^2\theta}$. This error arises as a result of the wrong sign in the expression for the retarded phase of the refracted wave. The author is indebted to V. L. Ginzburg for this information.

†In this case the expression for the transition radiation cannot be obtained for all media from Eq. (21) by simply reversing the particle velocity. This is immediately obvious from the fact that in the denominator we have the term $1-\beta\sqrt{\epsilon-\sin^2\theta}$, which vanishes at certain frequencies for transparent media.

Secondly, it can be shown in our case that for certain media and frequencies the above-mentioned pole is close to the saddle point or coincides with it. Then the part of the integrand in front of the exponential is not a slowly-varying function in the region of the saddle point and cannot be taken out from under the integral sign as in the preceding section. It can be shown that such a situation occurs if $\epsilon''(\omega') < \sqrt{c/\omega' R}$ for the frequency ranges $\omega' \pm \Delta\omega'$ where ω' and $\Delta\omega'$ are defined by the relations

$$\beta^2 \varepsilon'(\omega') - 1 = \beta^2 \sin^2\theta \text{ and } \Delta\omega' \sim \sqrt{c/\omega' R}\,(d\varepsilon/d\omega')^{-1}.$$

Since the increment $\Delta\omega'$ is small compared with ω', for the latter case we obtain a formula which refers only to the frequency ω'.

For simplicity let us assume that $\epsilon'' = 0$. Then, from (25) it is obvious that the pole adds to the integral if

$$0 \leqslant \beta^2 \varepsilon' - 1 \leqslant \beta^2 \sin^2\theta. \tag{26}$$

Then we have

$$E'_{2\rho} = \frac{e\beta^2}{\pi v R} \int\limits_{-\infty}^{+\infty} \frac{\sin\theta\cos\theta}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}} \left\{ \frac{\varepsilon + \beta\sqrt{\varepsilon - \sin^2\theta}}{1 - \beta^2\cos^2\theta} - \frac{1}{1 - \beta\sqrt{\varepsilon - \sin^2\theta}} \right\} e^{i\omega(R/c - t)}\, d\omega + \frac{2e}{v\sqrt{2\pi R\sin\theta\cdot v}}$$
$$\times \int\limits_{-\infty}^{+\infty} \frac{\sqrt{1 + \beta^2(1 - \varepsilon')}}{1 + \varepsilon\sqrt{1 + \beta^2(1 - \varepsilon')}} (\beta^2\varepsilon' - 1)^{1/4} \sqrt{-i\omega}\, \exp\left\{ i\left(-\omega t + \frac{\omega}{v}\sqrt{\beta^2\varepsilon' - 1}\, R\sin\theta + \frac{\omega}{v}\sqrt{1 + \beta^2(1 - \varepsilon')}\, R\cos\theta \right) \right\} d\omega. \tag{27}$$

The first term in this expression is to be associated with the transition radiation since the second term appears only if the condition in (26) is satisfied, yielding the Cerenkov radiation. The exponential term in the integrand of this term shows that the Cerenkov field of frequency ω is propagated at an angle $\vartheta(\omega)$ to the direction of motion where $\vartheta(\omega)$ is defined by

$$\sin\vartheta(\omega) = \sqrt{\beta^2\varepsilon'(\omega) - 1}/\beta.$$

This result can be understood on the basis of the following simple observations. Cerenkov radiation of frequency ω moves at an angle given by $\cos\theta' = 1/\beta\sqrt{\epsilon'(\omega)}$ in the first medium. If now we apply the law of refraction, the angle $\vartheta(\omega)$ is given by the above expression. The left-hand part of the double inequality in (26) is the condition for the production of Cerenkov radiation in the first medium. The right-hand part, however, which now can be written in the form $\vartheta(\omega) \leq \theta$, indicates that the field at frequency ω moving at an angle $\vartheta(\omega)$ can be seen at angles larger than or equal to $\vartheta(\omega)$.

Using similar arguments, we can obtain expressions for E'_{2n} and $H'_{2\varphi}$. From Eq. (27) it is obvious that the transition radiation and the Cerenkov radiation do not interfere. The total amount of transition radiation emitted during the time of flight of the particle, in the solid angle $d\Omega = \sin\theta\, d\theta\, d\varphi$, is

$$\frac{dW_{\text{trans.}}}{d\Omega} = \frac{ce^2\sin^2\theta\cos^2\theta}{\pi^2 v^2} \frac{\beta}{(1 - \beta^2\cos^2\theta)^2} \int\limits_0^\infty \left| \frac{(\varepsilon - 1)(1 - \beta^2 - \beta\sqrt{\varepsilon - \sin^2\theta})}{(\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta})(1 - \beta\sqrt{\varepsilon - \sin^2\theta})} \right|^2 d\omega. \tag{28}$$

In the case of transparent media, the integration in the last formula does not extend over frequencies which lie in the region $\omega' \pm \Delta\omega'$.

For the Cerenkov radiation, we compute the Poynting vector flux through the annular area ρ, $\rho + d\rho$ during the time of flight of the particle:

$$\frac{dW_{\text{trans.}}}{d\rho} = \frac{c}{4\pi} \int\limits_{-\infty}^{+\infty} dt E'_2 H'_{2\varphi} \cos\vartheta(\omega)\, 2\pi\rho = \frac{4e^2}{v^2} \int\limits_0^\infty \frac{\sqrt{(\beta^2\varepsilon' - 1)(1 + \beta^2(1 - \varepsilon'))}}{(1 + \varepsilon\sqrt{1 + \beta^2(1 - \varepsilon')})^2}\, \omega\, d\omega. \tag{29}$$

The integration in the last integral is performed only over those frequencies for which the condition in (26) is satisfied.

When $\omega = \omega'$ and $\epsilon'' = 0$, the saddle point is simultaneously a pole of the integrand, which is traversed from below. The integral along the line of descent is divided into the integral (in the sense of the principle value) and the half the residue at the pole. As a result, for frequencies close to ω' the following formula applies:

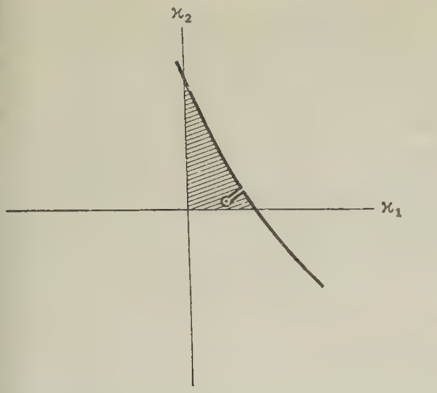

FIG. 4

$$\frac{dW_{\text{trans}}}{d\Omega} = \frac{ce^2 \sin^2\theta \cos^2\theta}{\pi^2 v^2} \frac{\beta^4}{(1-\beta^2\cos^2\theta)^2} \int \left(\frac{\varepsilon + 1}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}} \right)^2 d\omega. \quad (30)$$

The Poynting vector associated with the Cerenkov radiation also changes but only at the individual points $\omega = \omega'$.

We may note that here, just as in the preceding case, there is a region in which transition-radiation wave field is formed.

In conclusion the author wishes to express his gratitude to I. I. Gol'dman for a number of interesting discussions.

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EMISSION OF PARTICLES FROM EXCITED NUCLEI

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A formula for the dependence of the cross-section of a nuclear reaction involving the emission of several particles on the energy of incident nuclei has been obtained using the statistical theory of the nucleus and Bohr's concept of nuclear reactions. The formula has been used to compute the cross-sections for nuclear reactions involving the emission of 1, 2, 3, or 4 neutrons from Bi_{83}^{209} and I_{53}^{127} when bombarded with protons, deuterons, or α particles. The dependence of the entropy of a nucleus on the excitation energy and mass number has been determined using the gas model of the nucleus. The results of the calculations agree satisfactorily with the experimental data.

IT is known that an excited nucleus emits particles (n, p, α, γ, etc.) in transition to the ground state, the energy distribution of the particles being approximately Maxwellian (as follows, for example, from the evaporation model of the decay[1]). Accordingly, the mean energy carried away by an emitted particle is much smaller than the excitation energy of the nucleus.[2] If the excitation energy is sufficiently large the nucleus cannot, therefore, return to the ground state emitting a single particle.

Both theory[1] and experiment[3,4] show that the probability of emission of a single particle decreases with increasing excitation energy of the nucleus. Consequently, in transition to the ground state such a nucleus will emit several particles, the number of which depends on the initial excitation energy and on the binding energy of the particles in corresponding nuclei.

In the present note we calculate approximately, as a function of the excitation energy, the probability that a certain residual nucleus will be formed after consecutive emission of several (k) particles by the excited nucleus. Since excited nuclei are obtained by bombarding the target nucleus A with another nucleus a, it is easy to find the dependence of the investigated probability on the energies of the incident nuclei.

Consider the following reaction:

$$(A+a)\to(A+a)^*\to\left(A+a-\sum_{i=1}^{k} b_i\right)+\sum_{i=1}^{k}(b_i), \tag{1}$$

where (X) denotes a nucleus with mass number X. Since, according to Bohr, reaction (1) consists of two independent processes (formation of the compound nucleus $(A + a)^*$, and its decay), the cross-section for such a reaction can be written as[1]

$$\sigma_k=\sigma_c(a)\,\overline{\eta}_k, \tag{2}$$

where $\sigma_c(a)$ is the cross-section for the formation of a compound nucleus by the nuclei A and a, and $\overline{\eta}_k$ is the probability of the decay of the compound nucleus into final products.

The cross-section for the formation of the compound nucleus can be determined by means of the classical formula

$$\sigma_c(a)=\pi R_a^2\begin{cases}0, & E_a<B_a;\\ 1-B_a/E_a, & E_a\geqslant B_a,\end{cases}$$

where $R_a = r_0 A^{1/3} + \rho_a$, E_a is the energy of the incident particle in the center-of-mass system of A and a, and B_a is the Coulomb barrier, given by the formula

$$B_a=Z_A Z_a e^2/R_a. \tag{3}$$

Quantum-mechanical calculations show, however, that $\sigma_c(a)$ is different from zero even for $E_a = B_a$, when it equals[5]

$$\sigma_c(a)\approx 0.81 g^{-2/3}\pi R_a^2. \tag{4}$$

It is therefore possible to determine the cross-section for the formation of a compound nucleus by charged particles, with sufficient accuracy, in the following way ($B_n = 0$ for neutrons):

$$\sigma_c(a)\approx\pi R_a^2\left(1-(1-\alpha_a)B_a/E_a\right),\quad E_a\geqslant B_a; \tag{5}$$

$$\sigma_c(a)\approx\pi R_a^2\alpha_a\exp\{-2g\,f(E_a/B_a)\},\quad E_a<B_a, \tag{6}$$

where

$$f(x)=x^{-1/2}\arccos x^{1/2}-(1-x)^{1/2};\qquad \alpha_a=0{,}81g^{-2/3};\quad g=(2M_a Z_a Z_A e^2 R_a)^{1/2}/\hbar.$$

The cross-sections for reverse reactions, necessary for finding η_k, will be computed using formula (5) only, which considerably simplifies the calculation.

Formulae for the probability η_k of decay of the compound nucleus with emission of 1, 2, 3, etc. particles are given by various authors.[1,6-9] The formulae, however, have been calculated for the case when the compound nucleus, after emitting k_k particles, may be found in any state compatible with the law of the conservation of energy. As a rule, the values of η_k, calculated according to those expressions, either increase or decrease very slowly with the excitation energy, leading to a large discrepancy between the theoretical and experimental cross-sections.[9] This is connected with the fact that after emitting k particles, the residual nucleus can emit the next (k + 1)-th particle if the excitation energy is larger than the binding energy of any particle in the same nucleus. (Here, as in the following, we neglect the probability of γ emission since this process, on one hand, does not change the residual nucleus and, on the other, is very small compared with the probability of particle emission, when the excitation energy is larger than the binding energy.) In the following, therefore, we shall consider only the case when the nucleus, after emitting k particles, is in a state with excitation energy smaller than the binding energy of any particle in it. The probability of such a process is equal to the difference of the total emission probabilities of the k-th and the (k + 1)-th particle, i.e.,

$$\eta_k = I_k\left(U_a + \sum_{i=1}^{k} \varepsilon_{b_i}\right) - \sum_{\nu} I_{k+1}\left(U_a + \sum_{i=1}^{k} \varepsilon_{b_i} + \varepsilon_{b_{k+1,\nu}}\right), \tag{7}$$

where ϵ_{b_i} are the binding energies of particles b_i in the corresponding nuclei $A + a - b_1 - \ldots - b_{i-1}$, and $U_a = E_a - \epsilon_a$ is the excitation energy of the nucleus $(A + a)$. Summation over ν in the second term corresponds to the fact that the (k + 1)-th particle may be a neutron, a proton, etc. ($b_{k+1,\,\nu} = n; p \ldots$).

If

$$U_a < -\sum_{i=1}^{k} \varepsilon_{b_i} - \varepsilon_{b_{k+1,\,\nu}} \quad (I_{k+1} = 0),$$

then Eq. (7) coincides with the expressions given in Refs. 1, 6 – 9.

Equation (7) determines the probability of formation of the residual nucleus $(A + a - b_1 - \ldots - b_k)$ as the result of consecutive emission of the particles $b_1, b_2, \ldots, b_k$ from the nucleus $(A + a)$ in a given order. The same residual nucleus can be formed by emission of the same k particles in a different order or by emission of entire complexes of nuclei. For the description of a nuclear reaction it is therefore necessary to average Eq. (7) over all permutations of the k particles and then to carry out summation over all possible groupings of the emitted elementary particles, i.e.,

$$\bar{\eta} = \sum_{\text{group}} \frac{1}{k!} \sum_{(k)} (k)\, \eta_k, \tag{8}$$

where (k) denotes permutation; $\sum (k) = k!$

The cross-sections calculated by means of formulae (5), (6), and (8) can be compared with experimental data and used to compute the yield of various isotopes.

As an example, let us consider the following reaction:

$$\mathrm{Cu}_{29}^{63} + p \to (\mathrm{Zn}_{30}^{64})^* \to \mathrm{Cu}_{29}^{62} + \begin{cases} n_1 + p_2 \\ p_1 + n_2\,. \\ d_1 \end{cases}$$

Accordingly, the formation probability of the nucleus Cu_{29}^{62} will be

$$\bar{\eta} = [\eta_2(n_1;\, p_2) + \eta_2(p_1; n_2)]/2! + \eta_1(d_1)\,/\,1!$$

In order to calculate $I_k(x)$ we make use of the following equation:[9]

$$I_k(x) = \int_0^{x} f_1(E_{b_1})\, dE_{b_1} \int_0^{x - E_{b_1}} f_2(E_{b_2})\, dE_{b_2} \ldots dE_{b_k} \tag{9}$$

where $f_i(E_{b_i})\, dE_{b_i}$ is the relative differential probability of the i-th emission process, as given by the formula[1]

$$f_i(E_{b_i})\, dE_{b_i} = b_i \gamma_{b_i} \sigma_c(b_i)\, E_{b_i} \omega_i\, dE_{b_i} / \sum_{\nu} \int^{E_{\nu\max}} f_i(E_{b_{i\nu}})\, dE_{b_{i\nu}}, \tag{10}$$

where b_i, γ_{b_i}, and E_{b_i} are, respectively, the mass number, statistical weight, and kinetic energy of the emitted particle; $\sigma_c(b_i)$ is the production cross-section for the formation of a compound nucleus from the nuclei $(A + a - b_1 - \ldots - b_i)$ and (b_i). We shall use the following approximate expression for $\sigma_c(b_i)$ [cf. Eq. (5)]:

$$\sigma_c(b_i) = \pi R_{b_i}^2 \begin{cases} 1 - (1 - \alpha_{b_i})\, B_{b_i}/E_{b_i}; & E_{b_i} \geqslant (1 - \alpha_{b_i})\, B_{b_i}; \\ 0 & ; \; E_{b_i} < (1 - \alpha_{b_i})\, B_{b_i}. \end{cases} \tag{11}$$

ω_i is the level density in the nucleus $(A + a - b_1 - \ldots - b_i)$, given by the following formula:

$$\omega_i \sim (2J_i + 1)\, e^{S_i}, \tag{12}$$

where J_i is the spin of the stable or metastable state of the nucleus $(A + a - b_1 - \ldots - b_i)$, and S_i is its entropy:

$$S_i = S\Big(U_a + \sum_{\rho=1}^{i} (\varepsilon_{b_\rho} - E_{b_\rho});\ A + a - \sum_{\rho-1}^{i} b_\rho\Big). \tag{13}$$

Taking Eqs. (11) and (12) into account and introducing the notation

$$y_i = E_{b_i} - (1 - \alpha_{b_i}) B_{b_i},\ W_{b_i} = -\varepsilon_{b_i} + (1 - \alpha_{b_i}) B_{b_i}, \quad g_i = b_i \gamma_{b_i} R^2_{b_i} (2J_i + 1), \tag{14}$$

we obtain

$$f_i(y_i)\, dy_i = g_i y_i \exp\{S_i\}\, dy_i / \sum_{\nu} \int_0^{y_{\nu\max}} f_{i\nu}(y_i)\, dy_i, \tag{15}$$

where

$$S_i = S\Big(U_a - \sum_{\rho=1}^{i} (W_{b_\rho} + y_\rho);\ A + a - \sum_{\rho=1}^{i} b_\rho\Big); \quad y^{\nu}_{\max} = U_a - \sum_{\rho=1}^{i-1} (W_{b_\rho} + y_\rho) - W_{b_{i,\nu}}. \tag{16}$$

Consequently, the expression for η_k is

$$\eta_k = I_k(U_a - W_k) - \sum_{\nu} I_{k+1}(U_a - W_k - W_{b_{k+1,\nu}}), \tag{17}$$

where $I_n(x)$ is given by (9) and (15); the quantity $W_k = \sum_{i=1}^{k} W_{b_i}$ exceeds the threshold of emission of k particles from the excited nucleus, since we are determining the cross-sections for the reverse reactions induced by the charged particles by Eq. (11). In the case of neutron emission, W_k is exactly equal to the threshold. If both charged particles and neutrons are emitted, W_k represents excitation energy for which the yield of a given reaction becomes significant. In that case, W_k will be called the effective threshold, equal to $W_k + W_{b_{k+1,\nu}}$ for any $(k+1)$-th emission process.

In order to obtain the dependence of η_k on U_a, a k-fold integration of Eq. (9) is necessary. It can be seen from Eqs. (9) and (15) that the inner integrals depend also on U_a, which makes the computation (numerical integration) difficult. In order to simplify the calculation, it is convenient to make first the following substitution:

$$U_a - \sum_{\rho=1}^{i} (W_{b_\rho} + y_\rho) = t_i. \tag{18}$$

The calculation of $I_k(x)$ presents no difficulties when the dependence of the entropy of the nucleus on the excitation energy and on the mass number is known. If the results are applied to the simplest nuclear reactions, where neutrons are emitted from heavy nuclei $(A > 100)$, it is possible to make several approximations.

First, it is possible to neglect the emission of protons, α particles, etc. We have then, since the last inner integral in the expression for $I_k(x)$ equals unity,

$$\bar{\eta}^n_k \approx I^n_{k-1}(U_a - W_k) - I^n_k(U_a - W_{k+1}), \tag{19}$$

where W_k and W_{k+1} are the thresholds of emission of the k-th and $(k+1)$-th neutron from the excited nucleus.

Taking it into account that $\eta_k \neq 0$ only when $U_a > W_k$ and that

$$\sum_{\rho=1}^{i} W_{b_\rho} < W_k;\ \sum_{\rho=1}^{i} b_\rho < A + a, \tag{20}$$

we obtain the following approximation for S_i:

$$S_i \approx S_a - \frac{\partial S_a}{\partial A}\sum_{\rho=1}^{i} b_\rho - \frac{\partial S_a}{\partial U_a}\sum_{\rho=1}^{i}(W_{b_\rho} + y_\rho). \tag{21}$$

Substituting (21) into (15) and then into (9), and taking it into account that $(\partial S_a/\partial U)y_{\nu\ \max} \gg 1$, we obtain, after a few transformations,

$$I_k^n(x) \approx P_k\left(\frac{\partial S_a}{\partial U_a}x\right), \quad P_k(y) = \frac{1}{(2k-1)!}\int_0^y t^{2k-1}e^{-t}\,dt. \tag{22}$$

The functions $P_k(y)$ are tabulated[10] for various values of k and y. Graphs of the functions $P_k(y)$ for four values of k (indicated on the curves) and for $0 \le y \le 15$ are given in Fig. 1.

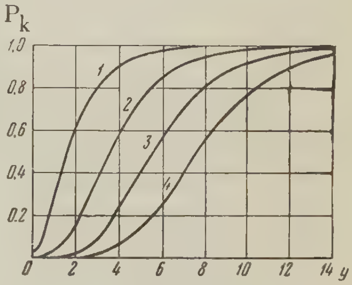

FIG. 1. Graphs of the functions $P_k(y)$.

The dependence of the entropy on the excitation energy and mass number can be obtained by assuming a specific nuclear model. Using the results of the gas model, we have[6]

$$S_a \approx \pi\sqrt{(A+a)U_a/33}, \qquad U_a = E_a - \varepsilon_a. \tag{23}$$

The cross-section for the reaction $(A + a) \rightarrow (A + a - kn) + k(n)$ is, therefore,

$$\sigma_{kn}(E_a) \approx \sigma_c(a)\,[P_{k-1}(y_k) - P_k(y_{k+1})], \tag{24}$$

where

$$y_k = \frac{\pi}{2}\sqrt{\frac{A+a}{33}}\left(E_a - \varepsilon_a - W_k\right)\Big/\sqrt{E_a - \varepsilon_a},$$

E_a is the energy of the incident particle in the center-of-mass system, and ϵ_a is the binding energy of this particle in the nucleus $(A + a)$.

The cross-sections for reactions accompanied by neutron emission have been calculated according to the formulae (24), (5), and (6), for incident protons, deuterons, and α particles. Certain thresholds W_k necessary for the calculation of theoretical curves have been determined by comparison with experimental cross-sections, mainly from their ratios, in order to exclude the cross-section for the compound nucleus formation. A direct determination of the thresholds from mass values is difficult, especially in the case of emission of two, three, etc. neutrons, since the masses of corresponding isotopes are known to an accuracy of 1 — 2 Mev. In addition, the following radii R_a have been assumed for the interactions between the incident particles and the nuclei:[1]

$$R_d \approx R_p = 1.5A^{1/3}\cdot 10^{-13}\ \text{cm}; \qquad R_\alpha = (1.5A^{1/3} + 1{,}2)\cdot 10^{-13}\ \text{cm}.$$

The interaction radii are rather arbitrary and, as a rule, yield an overvalued geometrical cross-section (by ~ 10%). However, since the production cross-section for a nucleus as calculated from Eqs. (5) and (6) is only approximate, it is not necessary to know R_a very accurately. Moreover, the dependence of the reaction cross-section on the energy of the incident particles will not change significantly since all cross-sections are calculated for $E_a \gtrsim 0.5\,B_a$, in which region the cross-section of compound nucleus formation depends relatively weakly on E_a and is not very sensitive to small variations in R_a.

The results of the calculations are shown in Figs. 2 and 3. It can be seen that the cross-sections obtained with Eq. (24) are in good agreement with the experimental data. The cross-section for the (d, n) reaction has not been calculated, since this process is mainly due to deuteron stripping and to splitting of the latter in the Coulomb field of the nucleus. The cross-section for the (α, n) reaction is not shown, in view of its smallness.

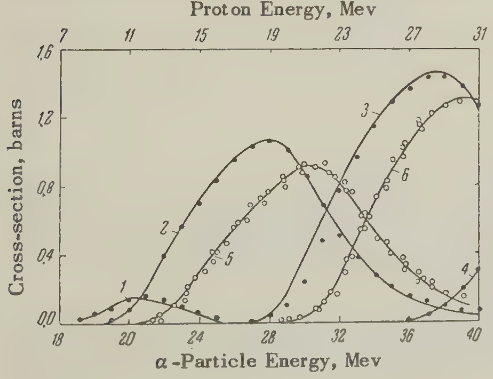

FIG. 2. Dependence of the Bi^{209}_{83} reaction cross-section on the energy of incident particles: 1 — (p, n); 2 — (p, 2n); 3 — (p, 3n); 4 — (p, 4n); 5 — $(\alpha, 2n)$; 6 — $(\alpha, 3n)$. The solid curves are drawn according to Eq. (24); ● — data of Ref. 4; ○ — data of Ref. 3.

In conclusion, the author wishes to express his gratitude to M. M. Agrest for valuable advice and discussion.

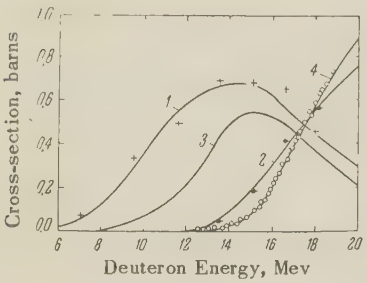

FIG. 3. Dependence of the Bi_{83}^{209} reaction cross-section on the deuteron energy: 1 — (d, 2n); 2 — (d, 3n); and of the I_{53}^{127} cross-section: 3 — (d, 2n); 4 — (d, 3n). Solid curves drawn according to Eq. (24); +, ● — data of Ref. 11; ○ — data of Ref. 3.

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SOME PROBLEMS OF MAGNETOGASDYNAMICS WITH ACCOUNT OF FINITE CONDUCTIVITY

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It is shown that if finite conductivity is taken into account, the equations of magnetogasdynamics become parabolically degenerate. The set of equations is replaced by an approximate but completely hyperbolic set, for which the characteristics are found. It is shown that the equations of a stationary one-dimensional flow have a singularity where the flow velocity is equal to the local sound velocity. Conditions of the transition of the flow velocity through this critical value under the action of a magnetic field have been studied. Small oscillations in a conducting medium, shock waves, and the structure of the shock are investigated.

THE magnetogasdynamics of an ideally conducting medium have been sufficiently studied. Types of vibration,[1,2] shock waves,[3,4] and their structure[5-7] have been investigated; one-dimensional motions have been studied,[8] where the characteristics were found for a system of equations and the particular (Reinmann) solution was found for arbitrary isentropy.

Taking the finite conductivity into account greatly complicates the equation by introducing new nonlinearities, raising the order of the system, and changing its character. As will be shown below, the system of equations changes from completely hyperbolic to parabolically degenerate. Finally, account of the Joule heat makes the motion essentially non-isentropic.

Small vibrations were first studied;[9] then, the possible types of arbitrary vibrations of the medium[11,12] were treated on the basis of the general wave equation.[10] Recently, Staniukovich found the particular solution by a perturbation method[13] for the case in which the conductivity is very large but finite.

In the present paper, several problems of one-dimensional motion are analyzed for a medium possessing arbitrary conductivity.

1. EQUATIONS EMPLOYED

If $\mathbf{v} \perp \mathbf{H} \perp \mathbf{E}$, where $\mathbf{v} = (u, 0, 0)$, $\mathbf{H} = (0, H, 0)$, and $\mathbf{E} = (0, 0, E)$, then the system of equations of magnetogasdynamics has, for the one-dimensional case, the form:[13]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{h}{\rho}\left(\frac{\partial h}{\partial x} - \frac{\partial e}{cdt}\right) = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = (\gamma - 1)\varkappa\left(\frac{\partial h}{\partial x} - \frac{\partial e}{c\partial t}\right)^2, \quad \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = \frac{\partial}{\partial x}\left[\varkappa\left(\frac{\partial h}{\partial x} - \frac{\partial e}{c\partial t}\right)\right], \quad \frac{\partial e}{\partial x} = \frac{\partial h}{c\partial t}. \tag{1}$$

Here, in addition to the usual notation, we have used

$$h = H/\sqrt{4\pi}; \quad e = E/\sqrt{4\pi}; \quad \gamma = c_p/c_v; \quad \varkappa = c^2/4\pi\sigma,$$

where c is the velocity of light, and σ is the conductivity of the medium, which we shall henceforth consider constant. The first equation of (1) is the equation of continuity, the second is the equation of motion, the third is the energy equation, the last two are Maxwell's equations with account of Ohm's law for a moving medium.

If we neglect the displacement current, then the system (1) is simplified:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{h^2}{2}\right) = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = (\gamma - 1)\varkappa\left(\frac{\partial h}{\partial x}\right)^2, \quad \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = \varkappa\frac{\partial^2 h}{\partial x^2}. \tag{2}$$

2. GENERAL PROPERTIES OF THE SYSTEM OF EQUATIONS

We shall show that the set of equations (2), together with (1), is parabolically degenerate.* We write down the set (2) as a set of five first-order equations for five unknown functions:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{hz}{\rho} = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} - (\gamma - 1)\varkappa z^2 = 0, \quad \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} - \varkappa\frac{\partial z}{\partial x} = 0, \quad \frac{\partial h}{\partial x} - z = 0. \tag{2a}$$

We convert the system (2a) to the characteristic form[14]

$$C(y) = \begin{vmatrix} y_1 + uy_2 & \rho & 0 & 0 & 0 \\ 0 & y_1 + uy_2 & 1/\rho & 0 & 0 \\ 0 & \gamma p & y_1 + uy_2 & 0 & 0 \\ 0 & h & 0 & y_1 + uy_2 & -\varkappa y_2 \\ 0 & 0 & 0 & y_2 & 0 \end{vmatrix}.$$

The system would be completely hyperbolic if the algebraic equation $C(y) = 0$ had five real, not necessarily different, roots. But we have the equation

$$y_2^2(y_1 + uy_2)\left[(y_1 + uy_2)^2 - \gamma p/\rho\right] = 0,$$

which has only three roots.

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*The possibility of parabolic degeneracy was pointed out to us by B. L. Rozhdestvenskii.

In similar fashion, if we look at the set (1), we then obtain a set of six equations for the six unknown functions. The corresponding characteristic equation $C = 0$ has only five roots for y_1.

The parabolic degeneracy is unfortunate in that it deprives us of the possibility of studying the system by the method of characteristics. However, we can replace the set (2) by an approximate set of the hyperbolic type. Let $\partial^2 h/\partial x^2$ be bounded everywhere. We then write

$$(\partial h / \partial x)^2 = b_1 \partial h / \partial x, \quad \partial^2 h / \partial x^2 = b_2 \partial h / \partial x,$$

where

$$b_1 = \overline{\frac{\partial h}{\partial x}}, \quad b_2 = \overline{\frac{\partial}{\partial x}\left(\ln \frac{\partial h}{\partial x}\right)}.$$

After such an averaging, we can write the set (2) in the form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \rho \frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{h}{\rho}\frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \gamma p \frac{\partial u}{\partial x} - (\gamma - 1) \varkappa b_1 \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} - \varkappa b_2 \frac{\partial h}{\partial x} = 0. \tag{2b}$$

It is not difficult to verify the fact that this system is completely hyperbolic. The characteristic relation for it has the form

$$\begin{vmatrix} u - \dot{x} & \rho & 0 & 0 \\ 0 & u - \dot{x} & 1/\rho & h/\rho \\ 0 & \gamma p & u - \dot{x} & -\varkappa b_1 (\gamma - 1) \\ 0 & h & 0 & u - \dot{x} - \varkappa b_2 \end{vmatrix} = 0.$$

Expanding the determinant, we first get

$$u - \dot{x} = 0; \quad dx/dt = u. \tag{3}$$

This is the equation of the flow line. Along it, the equation

$$dQ = T\, dS = (j^2 / \rho\sigma)\, dt, \tag{4}$$

is satisfied, since $dQ/dt = j^2/\rho\sigma$ is the influx of Joule heat per unit time. Furthermore, we get

$$(u - \dot{x})^2 = a_m^2 + \varkappa \frac{b_2 [(u - \dot{x})^2 - a_0^2] + h b_1 (\gamma - 1)/\rho}{u - \dot{x}},$$

where

$$a_m^2 = \gamma p / \rho + h^2/\rho = a_0^2 + V^2.$$

In the zeroth approximation, $\sigma \to \infty$, i.e., $\kappa \to 0$, and we get the well known result[8]

$$dx/dt = u \pm a_m. \tag{5}$$

If we take into account terms of first order of smallness in κ, we can compute

$$dx/dt = u \pm a_m \mp (\varkappa / 2a_m^2)\, [b_2 V^2 + h b_1 (\gamma - 1)/\rho]. \tag{6}$$

The equation

$$d\left(p + \frac{h^2}{2}\right) = \pm \rho a_m\, du \mp \frac{\varkappa}{a_m^2}\left[b_2 V^2 + \frac{h}{\rho} b_1 (\gamma - 1)\right]\left(a_m \frac{dh}{h} + \frac{du}{2}\right), \tag{7}$$

which is computed in the same approximation, is satisfied, along this sytem of characteristics.

It is evident that the velocity of transfer of the excitation in the gas diminishes with increasing κ, i.e., decreasing conductivity.

3. STATIONARY MOTIONS

In this case, the set (2) becomes a set of ordinary differential equations

$$\frac{d\rho u}{dx} = 0, \quad \rho u \frac{du}{dx} + \frac{d}{dx}\left(p + \frac{h^2}{2}\right) = 0, \quad u \frac{dp}{dx} + \gamma p \frac{du}{dx} = (\gamma - 1) \varkappa \left(\frac{dh}{dx}\right)^2, \quad \frac{dhu}{dx} = \varkappa \frac{d^2 h}{dx^2}. \tag{8}$$

This set has a series of integrals

$$\rho u = m, \quad mu + p + h^2 / 2 = J, \quad u\, dp / dx + \gamma p\, du / dx = (\gamma - 1) \varkappa (dh / dx)^2, \quad hu = \varkappa\, dh / dx + mb, \tag{9}$$

where m, J, and b are constant quantities. From (9) we can obtain a single second-order equation for h′:

$$h'' = \frac{h'}{h}\left[h' + \frac{mb}{\varkappa} + \frac{(\gamma h' + mb / \varkappa)\, h^3}{\gamma h (J - h^2 / 2) - (\gamma + 1)\, m\, (\varkappa h' + mb)}\right]. \tag{10}$$

It has a singularity where

$$\gamma h (J - h^2 / 2) - (\gamma + 1)\, m\, (\varkappa h' + mb) = 0.$$

Since

$$J - h^2 / 2 = mu + p, \quad \varkappa h' + mb = hu, \quad m = \rho u,$$

then at the singularity $u^2 = \gamma p/\rho$. Here too, as in ordinary gasdynamics, the flow velocity, which is equal to the local velocity of sound, has a critical value. It should be observed that this result is valid for any value of the conductivity, large or small, and also for values varying with distance, as is readily seen from (8) and (9). Morozov arrived at the same result by an entirely different method.[15]

The order of Eq. (10) can be lowered; it is not difficult to integrate it numerically, following all possible stationary flows. However, clarification of the behavior of quantities near the critical value of the velocity is also possible by direct consideration of the problem.

For stationary flow we can write down Bernoulli's equation

$$\frac{dp}{\rho} + \frac{du^2}{2} + \frac{dh^2}{2\rho} = 0. \tag{11}$$

Let the pressure be a function of the density and entropy, $p = p(\rho, S)$. Then, using well known thermodynamic relations, we get

$$dp = a_0^2\left[d\rho - \frac{T}{c_p}\left(\frac{\partial \rho}{\partial T}\right)_p dS\right]. \tag{12}$$

Comparing (11) and (12), we get, considering that $\rho u = m = \text{const}$,

$$\rho \frac{du^2}{2} + \frac{dh^2}{2} = a_0^2\left[\frac{\rho}{u}\, du + \frac{T}{c_p}\left(\frac{\partial \rho}{\partial T}\right)_p dS\right].$$

Introducing $dQ = TdS$ and $M = u/a_0$, we obtain, after a number of transformations,

$$(M^2 - 1)\frac{du}{u} = \left(\frac{\partial \rho}{\partial T}\right)_p \frac{dQ}{\rho c_p} - \frac{hdh}{\rho a_0^2}. \tag{13}$$

In our case, $dQ = (j^2/\rho\sigma)\, dt$ and $dt = dx/u$. Maxwell's equation gives

$$j = \frac{c}{\sqrt{4\pi}} \frac{dh}{dx}.$$

As a result,

$$dQ = (\varkappa / m)\, (dh / dx)^2\, dx. \tag{14}$$

Equation (13) is the standard form of the Vulis law of the inversion of action,[16] which states that identical conditions produce in a flow opposite effects for subsonic and supersonic flow. For example, contraction of the jet accelerates the subsonic flow and retards the supersonic. In ordinary gasdynamics, this fact is associated with changeover of the flow under the action of the excitations propagated in it with the velocity of sound.[16] In a conducting medium in a magnetic field, the excitations are propagated with a velocity greater than $a_0 = \sqrt{\gamma p/\rho}$. Why is a_0 exactly the critical velocity in our case?

If $u = a_0$, i.e., $M = 1$, then the left side of (13) is equal to zero. For this to happen, we must have in the critical cross section

$$dh/dx = 0. \tag{15}$$

If this is so, this field will not affect the motion in precisely the critical region of flow, since the corresponding terms in the equations disappear in this region. Therefore the excitations there ought to be propagated with the ordinary velocity of sound.

The right side of (13) also vanishes for

$$h = h_0 \exp(-x/\rho a_0^2 \alpha), \text{ where } \alpha = -(\partial\rho/\partial T)_p \varkappa/m,$$

But such a special value of the field has only an accidental relation to the critical point.

Let us investigate the conditions for the possibility of going through the velocity of sound.

For $M < 1$ and an accelerating flow, the right side of (13) must be less than zero. Let us write it in the form

$$d\Sigma = \left[\left(\frac{\partial\rho}{\partial T}\right)_p \frac{\varkappa}{m}\left(\frac{dh}{dx}\right)^2 - \frac{h}{\rho a_0^2}\frac{dh}{dx}\right]dx = -\left[\alpha\left(\frac{dh}{dx}\right)^2 + \frac{h}{\rho a_0^2}\frac{dh}{dx}\right]dx.$$

In our case of an ideal gas, $(\partial\rho/\partial T)_p = -\rho/T < 0$, therefore $\alpha > 0$; $d\Sigma < 0$, if $dh/dx > 0$ and if $h/\rho a_0^2 > -\alpha dh/dx$. Thus, if the field increases with x, or falls more slowly than $h_0 \exp(-x/\rho a_0^2 \alpha)$, then the subsonic flow is accelerated.

For $M > 1$, $d\Sigma > 0$, and for an accelerating supersonic flow, the field must fall more rapidly than $h_0 \exp(-x/\rho a_0^2 \alpha)$. In the opposite case, the sign of the effect does not change, and in a stationary mode the flow velocity cannot be greater than a_0. Thus, by passing definite currents through the gas, we can accelerate the ionized gas, i.e., we can construct a magnetic jet of a kind.

It is not difficult to obtain expressions analogous to (13) for p, ρ and h, and to carry through other detailed studies just as for ordinary flows.[16] The behavior of all the quantities close to the critical value is given in Fig. 1.

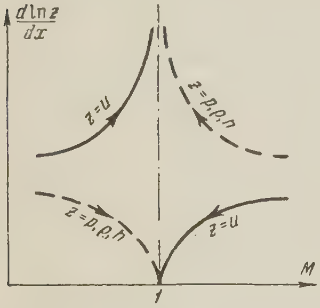

FIG. 1. Diagram of the current relationship. The direction of the process is shown by the arrows.

Although the critical velocity of flow, which is equal to the local sound velocity, does not depend explicitly on the field, it depends on the field implicitly, since ρ and h are connected with one another in a conducting medium. It is possible to find the dependence of the critical velocity a_* on the field for an ideally conducting medium and to estimate the dependence in our case also.

Let us transform the expression for the flow of energy (26a) [derived in Sec. 6] with the help of the relation $h/\rho = b = \text{const.}$ and the adiabatic of Poisson $\rho a_0^{-2/(\gamma-1)} = \text{const.} = A_1$. We get

$$\frac{u^2}{2} + \frac{a_0^2}{\gamma-1} + A_1 b^2 a_0^{2/(\gamma-1)} = \text{const} = \frac{a_{00}^2}{\gamma-1} + A_1 b^2 a_{00}^{2/(\gamma-1)}. \tag{16}$$

We introduce the parameter

$$\mu = A_1 b^2 a_{00}^{2(2-\gamma)/(\gamma-1)}.$$

For $\gamma = 5/3$, μ has the meaning of the square of the ratio of the Alfven velocity $V = h/\sqrt{\rho}$ to the sound velocity a_{00}, determined for the gas at rest. When the flow reaches the critical velocity, $u = a_*$. Introducing the ratio $x_* = a_*/a_{00}$, we rewrite (16) as

$$\frac{x_*^2}{2} + \frac{x_*^2}{\gamma-1} + \mu x_*^{2/(\gamma-1)} = \frac{1}{\gamma-1} + \mu. \tag{16$'$}$$

This equation determines the dependence of the critical velocity on the parameter μ, which characterizes the field. Let us represent (16′) in the form

$$\mu\left(1 - x_*^{2/(\gamma-1)}\right) + \frac{1}{\gamma-1}\left(1 - \frac{\gamma+1}{2}x_*^2\right) = 0. \tag{16$''$}$$

μ	x_*
0	0.866
0.0404	0.87
0.2854	0.89
0.6239	0.91
1.1494	0.93
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
∞	1.00

As μ changes from 0 to ∞, x_* changes from $\sqrt{2/(\gamma+1)}$ to 1. The dependence $x_* = x_*(\mu)$ for $\gamma = 5/3$ is shown in the table. For $\gamma = 1.4$, this dependence is still weaker: x_* changes from $\sqrt{5/6} = 0.9129$ to 1.

4. SMALL EXCITATIONS

Let u be a small quantity, and let $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $h = h_0 + h'$, where small quantities are denoted by a prime. In what follows, we shall omit the primes. The system (2) is linearized in the usual fashion:

$$\frac{\partial\rho}{\partial t} + \rho_0\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{h_0}{\rho_0}\frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + h_0\frac{\partial u}{\partial x} - \varkappa\frac{\partial^2 h}{\partial x^2} = 0. \tag{17}$$

Since the Joule losses are neglected in the linearization, the energy equation is equivalent to the adiabatic equation $p = \text{const}\cdot\rho^\gamma$. In this case, it is possible to make the following transformation:

$$\frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{\partial\rho}{\partial x} = a_0^2\frac{\partial\rho}{\partial x}.$$

We look for all quantities in (16) proportional to $\exp\{i(kx - \omega t)\}$. Eliminating $\rho/\rho_0 = ku/\omega$, we get

$$(k^2a_0^2/\omega - \omega)\,u + (kh_0/\rho_0)\,h = 0, \quad kh_0u - (\omega + ik^2\varkappa)\,h = 0.$$

As the condition for nontrivial solution of this system, we get the dispersion equation

$$k^4 - k^2\left(\frac{\omega^2}{a_0^2} - \frac{i\omega}{\varkappa}\frac{a_m^2}{a_0^2}\right) + \frac{i\omega}{\varkappa a_0^2} = 0. \tag{18}$$

Let us investigate the solution of Eq. (18) in two limiting cases:
(1) $\omega \gg a_m^2/\kappa$ — the case of a weakly conducting gas (κ large);
(2) $\omega \ll a_m^2/\kappa$ — the case of a strongly conducting gas (κ small).
With accuracy up to terms of first order of smallness, we have

$$\begin{aligned} k_1^+ &= (\omega/a_0) + (iV_0^2/2a_0\varkappa); & k_1^- &= (1+i)\sqrt{\omega/2\varkappa}, \\ k_2^+ &= (1+i)\sqrt{\omega/2\varkappa}\,(a_m/a_0); & k_2^- &= (\omega/a_m) + (i\omega^2\varkappa/2a_m^3). \end{aligned} \tag{19}$$

The wave numbers k_1^- and k_2^- correspond to the ordinary acoustic wave and a wave propagated with the effective magnetogasdynamical sound velocity[2,3,8]. The quantities k_1^- and k_2^- correspond to waves of the skin type, and are rapidly damped. The indices ± correspond to the sign chosen in front of the inner root of the solution of the biquadratic equation (18).

It is interesting that the ordinary acoustic wave and its analogue in the case of infinite conductivity belong to different frequency modes and never cross each other upon corresponding increase or decrease of the conductivity, but degenerate into waves of the skin type. It is true that for such a change of the magnitude of the conductivity, we pass through regions of strong absorption, where the wave and its velocity lose meaning, so that this result takes on, perhaps, a somewhat formal character.

An equation of the type (18) was earlier obtained by Anderson;[17] however, he assumed that a_m is close to a_0 (weak field), therefore, he did not obtain from his approximate solution that a_m and a_b belong to different frequency modes.

5. ANOTHER METHOD OF INVESTIGATION OF SMALL EXCITATIONS

By linearization of the system (17), we can obtain a single equation for the velocity potential φ, which is introduced by the equation $u = \partial\varphi/\partial x$:

$$\varkappa\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\varphi}{\partial t^2} - a_0^2\frac{\partial^2\varphi}{\partial x^2}\right) = \frac{\partial}{\partial t}\left(\frac{\partial^2\varphi}{\partial t^2} - a_m^2\frac{\partial^2\varphi}{\partial x^2}\right). \tag{20}$$

We note that the form of Eq. (20) is strongly reminiscent of the equation of thermal condictivity, which again bears witness to the parabolic degeneracy.

We seek the solution by the method of separation of variables:

$$\varphi = X(x)\,T(t).$$

Let

$$X^{IV} + k^2 X'' = 0, \quad X'' + k^2 X = 0.$$

Thence

$$X = B_1 \cos kx + B_2 \sin kx.$$

Then we get for T from Eq. (20):

$$\dddot{T} + a_m^2 k^2 \ddot{T} + k^2 \varkappa (\ddot{T} + a_0^2 k^2 T) = 0.$$

We seek a solution in the form $T = \exp(\alpha t)$. For the determination of α, we obtain the cubic equation

$$\alpha^3 + k^2 \varkappa (\alpha^2 + a_0^2 k^2) + a_m^2 k^2 \alpha = 0. \tag{21}$$

This equation, for arbitrary values of κ, has two complex and one real root.

We first consider the case of small κ. With accuracy up to terms of second order of smallness in κ,

$$\alpha_{1,2} = -\frac{\varkappa k^2}{2a_m^2} V_0^2 \pm i a_m k \left[1 - \frac{\varkappa^2 k^2}{8 a_m^4} V_0^2 \left(1 + \frac{3a_0^2}{2a_m^2}\right)\right] = -\varkappa M \pm iN, \qquad \alpha_3 = -\varkappa a_0^2 k^2 / a_m^2 = -\varkappa R$$

(the notation is self evident). As a result,

$$T = e^{-\varkappa M t} (A_1 \cos Nt + A_2 \sin Nt) + A_3 e^{-\varkappa R t}.$$

We can always write down a certain particular solution of Eq. (20) in the form

$$\overline{\varphi} = \varphi_0 \cos [N(t - \tau)] \cos [k(x - l)]. \tag{22}$$

To find k, we must solve some sort of actual boundary problem, for which we can construct a general solution in the form of the superposition of standing waves, summing all possible particular solutions of type (22). However, this is not necessary for the determination of the phase (or group) velocity of propagation of a small vibration. For example, we get for the phase velocity $v = N/k = \omega/k$,

$$v = a_m \left[1 - \frac{\varkappa^2 k^2}{8 a_m^4} V_0^2 \left(1 + \frac{3a_0^2}{2a_m^2}\right)\right]. \tag{23}$$

In the case $\sigma \to \infty$ we get $v \to a_m$, as was to have been expected.

In the limit $\sigma \to 0$, we have $\kappa \to \infty$ and from Eq. (21) [with accuracy up to terms of the second order of smallness in $1/\kappa$] we get $\alpha_{1,2} = \pm i a_0 k$, whence the phase velocity is $v = a_0$.

We shall not carry out any further investigation for obtaining skin waves.

6. SHOCK WAVES

For simplicity, let us consider only a rectilinear shock propagated with a velocity u_1 in a motionless, conducting gas located in a magnetic field h_1. The wavefront is parallel to the field.

The equations of stationary one-dimensional flow have the three integrals (9) of mass flow, momentum flux, and magnetic field flux:

$$\rho u = m, \quad mu + p + h^2/2 = J, \quad uh = \varkappa\, dh/dx + mb. \tag{24}$$

The Maxwell equations yield

$$e = \mathrm{const} = -\frac{u_1 h_1}{c}; \quad j = \frac{c}{\sqrt{4\pi}} \frac{dh}{dx}. \tag{25}$$

The equation of energy flux has the form:

$$m \frac{d}{dx}\left(i + \frac{u^2}{2}\right) = \sqrt{4\pi}\, je,$$

where i is the specific enthalpy. Substituting Eq. (25) and integrating, we get the integral of energy flux:

$$m(i+u^2/2)=mA-u_1h_1h, \text{ where } A=\text{const}. \tag{26}$$

We can rewrite Eq. (25) in the form

$$i+(u^2/2)+bh=A. \tag{26a}$$

The set of equations (24) and (26) completely determine the parameters of stationary flow.

It is now easy to write down the conditions on the front of the discontinuity:

$$\begin{gathered}\rho_1u_1=\rho_2u_2,\\ \rho_1u_1^2+p_1+h_1^2/2=\rho_2u_2^2+p_2+h_2^2/2,\\ i_1+(u_1^2/2)+bh_1=i_2+(u_2^2/2)+bh_2,\\ u_1h_1-(\varkappa dh/dx)_1=u_2h_2-(\varkappa dh/dx)_2.\end{gathered} \tag{27}$$

The index 1 denotes the parameters in front of the shock, the index 2 those behind the shock. If the conductivity is infinite, then $h_1/\rho_1=h/\rho=b=\text{const}$, $\kappa=0$, and the conditions (27) coincide with the known expressions on the front of the shock.[3]

It follows from the first two equations of (27) that

$$u_1^2=\frac{\rho_2}{\rho_1}\left[\frac{p_2-p_1}{\rho_2-\rho_1}+\frac{h_2^2-h_1^2}{2(\rho_2-\rho_1)}\right],\quad u_2^2=\frac{\rho_1}{\rho_2}\left[\frac{p_2-p_1}{\rho_2-\rho_1}+\frac{h_2^2-h_1^2}{2(\rho_2-\rho_1)}\right]. \tag{28}$$

For $p_2=p_1+\Delta p$, $\rho_2=\rho_1+\Delta\rho$, $h_2=h_1+\Delta h$, we shall have

$$u_1^2=u_2^2=a_*^2=(\Delta p/\Delta\rho)+(h\Delta h/\Delta\rho), \tag{29}$$

where a_* is the effective sound velocity in the medium under consideration. From the last equation of the system (24),

$$\frac{h}{\rho}=b+\frac{\varkappa}{m}\frac{dh}{dx}, \tag{30}$$

whence

$$\frac{\Delta h}{\Delta\rho}=b+\frac{\varkappa}{m}\frac{\Delta}{\Delta\rho}\left(\rho\frac{dh}{dx}\right);\quad a_*^2=a_0^2+\left(b+\frac{\varkappa}{m}\frac{dh}{dx}\right)^2\rho+\frac{\varkappa}{m}\rho^2\left(b+\frac{\varkappa}{m}\frac{dh}{dx}\right)\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right).$$

This expression, with the help of (30), can be put in the form

$$a_*^2=a_0^2+\frac{h^2}{\rho}+\frac{\varkappa}{m}h\rho\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right). \tag{31}$$

Since the flow takes place from right to left, then $dh/dx<0$ and

$$a_*^2<a_m^2=a_0^2+h^2/\rho. \tag{32}$$

We estimate the value of the contribution connected with the conductivity, relative to h^2/ρ. We have

$$\varkappa=c^2/4\pi\sigma=c^2m_e/4\pi ne^2\tau.$$

As will be shown later, change of the field takes place at a distance of the order of the mean free path $\ell=m_ea_1\sigma/ne^2$. Then

$$\frac{h\rho}{m}\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right)\approx-\frac{h^2}{u\rho l}.$$

The ratio in which we are interested will have a value of order $1/\Pi_e$, where $\Pi_e=(e^2/m_ec^2)n\ell^2$ is a dimensionless quantity, the so called "linear" electron.

Let us compute the change in the entropy for a weak discontinuity. In this case,

$$u_1=u_2=a_*;\quad h_2-h_1=\Delta h=(\varkappa/a_*)\,dh/dx.$$

From (14) we get

$$dS = \frac{dQ}{T} = \frac{\varkappa}{mT}\left(\frac{dh}{dx}\right)^2 dx; \quad S_2 - S_1 = \int\limits_0^L \frac{\varkappa}{mT}\left(\frac{dh}{dx}\right)^2 dx.$$

Obtaining the mean values T and dh/dx over the region of change of the quantities (0 to L), we get

$$S_2 - S_1 = \Delta S = (a_*^2 L/mT\varkappa)\,(\Delta h)^2. \tag{33}$$

Thus, the change in the entropy is a quantity of second order of smallness in comparison with the change in the field.

7. STRUCTURE OF THE SHOCK WAVE

Although this question has already been studied in detail by Marshall[6,7] with consideration of viscosity and thermal conductivity, it occurred to us that it is of methodological and some practical interest to consider the weakening of the front under the action of conductivity alone.

Let us introduce the dimensionless variables

$$\nu = u/a_1; \quad \eta = p/p_1; \quad \chi = h/h_1, \text{ where } a_1 = \sqrt{\gamma p_1/\rho_1}.$$

We choose $\xi = x/\lambda$ as the dimensionless unit of length, where $\lambda = c^2/4\pi\sigma a_1 = \kappa/a_1$. The conditions (27) become in the non-dimensional form,

$$k\nu + \eta + Q_1\chi^2 = J', \quad \frac{\gamma}{1-\gamma}\nu\eta + k\frac{\nu^2}{2} + 2\nu_1 Q_1\chi = A', \quad \frac{d\chi}{d\xi} = \nu\chi - \nu_1, \tag{34}$$

where use is made of the fact that $i = (\gamma p/\rho)/(\gamma - 1)$ for an ideal gas; $k = ma_1/p_1$; $J' = J/p_1$; $A' = Am/p_1a_1$; $Q_1 = h_1^2/2p_1$. From the first two equations of (34), we have

$$\nu = \nu(\chi) = \frac{\gamma}{(\gamma+1)k}(J' - Q_1\chi^2) + \left\{\left[\frac{\gamma}{(\gamma+1)k}(J' - Q_1\chi^2)\right]^2 - \frac{2(\gamma-1)}{k(\gamma+1)}(A' - 2\nu_1 Q_1\chi)\right\}^{1/2}. \tag{35}$$

As computations have shown, the minus sign in front of the root has no meaning. From the last equation of (34), we now find $\xi - \xi_0$ by quadrature. This integral was computed for the case of monatomic deuterium at T = 20,000°K, p_1 = 20,000 g/cm-sec^2 (about 15 mm Hg), H_1 = 689 oersted, and $\nu_1 = 2$. As is seen from Fig. 2, the change in all the quantities takes place over a distance on the order of two units. If we assume that in this case, $\sigma \approx 10^{14}$ cgs units, then the change will take place over a distance of approximately two mean free path lengths.

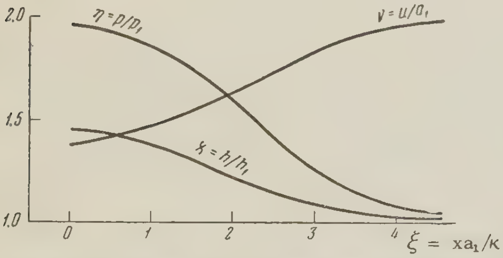

FIG. 2. Profile of the jump. The shock wave moves from left to right.

If $\lambda \to 0$, then $x = \xi\lambda \to 0$. Thus, in the case of very high conductivity, we get a sharp discontinuity. It should be pointed out that in fact a large thermal conductivity also corresponds to very high electric conductivity; therefore the jump will be strongly diffused by reason of the thermal conductivity.[7]

If the conductivity is low, the field will change slowly at large distances, on the order of κ/a_1. If the jump is strong, then it is clear that the field should have no effect on the shock wave. We shall analyze this in detail. We write down the condition for conservation of momentum for the specific volume $V = 1/\rho$:

$$p + m^2V + h^2/2 = p_1 + m^2V_1 + h_1^2/2. \tag{36}$$

We differentiate this equation with respect to V:

$$\left(\frac{dp}{dT}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T + m^2 + h\,\frac{dh}{dV} = 0. \tag{37}$$

$(\partial p/\partial T)_V > 0$ for all gasses; $dT/dV < 0$, since the gas in the shock wave is heated in the compression; $dh/dV < 0$, since $dh/d\rho > 0$. On side 1 we have

$$m^2 > -\left[\left(\frac{\partial p}{\partial T}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T\right]_1, \tag{38}$$

since $u_1 > a_1$, and to satisfy Eq. (37), we must have $dh/dV < 0$, which is to be expected. If the intensity of the shock wave is so large that

$$m^2 < -\left[\left(\frac{\partial p}{\partial T}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T\right]_2, \tag{39}$$

then to satisfy condition (37), we must have $dh/dV > 0$; this would mean that upon compression from V_1 to V_2 the field passes through a maximum, and thereafter falls to the value h_2; this cannot be. The real picture is the following; at first, in the compression from the volume V_1 to some intermediate volume V', a slow continuous change of the quantities takes place. Here $dh/dv < 0$, and the field changes from h_1 to h_2. After the shock, a compression takes place from V' to V_2. The field in this case does not change, while the velocity, pressure, and temperature undergo a jump. If condition (39) is not satisfied, then there will be a slow continuous change of all quantities to a distance of the order κ/a_1. This phenomenon is very much like an isothermal jump;[18] therefore, by analogy, it ought to be called an isomagnetic jump.

The condition (39), as is easily seen, is equivalent to $u_2 < a_2$. It is possible to show[6] that it reduces to

$$Q_1\left\{2(2-\gamma)\frac{u_1}{u_2} + 2\gamma - \gamma(\gamma-1)\left(\frac{u_1}{u_2}-1\right)^3\right\} < \gamma(\gamma+1)\left(\frac{u_1}{u_2}-1\right). \tag{40}$$

If $u_1/u_2 > 2.64$ for $\gamma = 5/3$, then the left side of (40) will be less than zero and the jump will always occur. For smaller intensities the fact as to whether a discontinuity is approached or not will depend on the magnitude of the field.

When Marshall, by numerical integration of the differential equations for a certain example, obtained this picture, it appeared strange to him that all the change of the field took place in front of the jump. We see here that it could not be otherwise. In this lies the essence of the isomagnetic jump.

In conclusion, we express our deep appreciation to Academician M. A. Leontovich for his constant interest in the work and for many useful discussions.

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 6 JUNE, 1958

NONLINEAR EFFECTS OF THE INTERACTION OF RESONANCE FIELDS IN THE MOLECULAR GENERATOR AND AMPLIFIER

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The position of the possible frequencies of generation and amplification, with saturation by an auxiliary field taken into account, is investigated on the basis of an analysis of the polarizability of a quantum system situated in two resonance fields. It is shown that an amplifier or a generator can operate at two frequencies which depend on the amplitude as well as on the frequency of the auxiliary field.

BASOV and Prokhorov[1] have considered a molecular generator and amplifier that does not utilize a molecular beam. In this case an auxiliary field of higher frequency is required for obtaining active molecules.

In contrast to the generator with a beam, where, with any kind of excitation (state selection), only the number of active molecules obtained plays a role, in generators and amplifiers with an auxiliary field the molecules must be situated simultaneously in two resonance fields with a common resonance level. In this case the auxiliary field will affect the shapes of the lines and the positions of the resonant frequencies of the transition used. Similar effects were observed by Burgess and Norberg[2] in the hyperfine spectrum of the $(SO_3)_2NO^{--}$ radical and are easily explained on the basis of the quantum theory of dispersion.

Let two resonance fields with frequencies ω_1 and ω_2 act simultaneously on a quantum system with a nonequidistant discrete spectrum; let these frequencies be close to the transition frequencies ω_{mn} and ω_{mq} between any levels of the system (these levels will be called resonance levels), under the condition that one of the resonance levels be common to both fields.

We shall give an expression for the Fourier component of the average dipole moment of a gas, in which τ is the average time between collisions of a molecule, obtained under the usual assumptions,[3] but in the presence of two fields with frequencies

$$|\omega_1-\omega_{mn}|\lesssim 1/\tau, \quad |\omega_2-\omega_{mq}|\lesssim 1/\tau$$

and a common resonance level m. The perturbed Hamiltonian is $\hat{H}(t)=\hat{H}_0+\hbar\hat{V}(t)$, where the perturbation has the form

$$\hat{V}(t)=\hat{\Phi}^{\pm}e^{\mp i\omega_1 t}+\hat{\Psi}^{\pm}e^{\mp i\omega_2 t}$$

(summation over ±). The resonance part of the density matrix $\hat{D}$ is determined from the equation

$$\left(\frac{\partial}{\partial t}+i\omega_{ab}+\frac{1}{\tau}\right)D_{ab}+i\sum_k\left(V_{ak}(t)\,D_{kb}-D_{ak}V_{kb}(t)\right)=\frac{\rho_a^0-\rho_b^0}{\omega_{ab}}\left(\frac{\partial\hat{V}}{\partial t}\right)_{ab}, \qquad (1)$$

where

$$\hat{\rho}^0=\exp(-\hat{H}_0/kT)/\mathrm{Sp}\exp(-\hat{H}_0/kT)$$

is the equilibrium distribution function in the absence of a field. Near resonance with $|V|\gtrsim 1/\tau$ (strong field) it is necessary to take into account the second term in (1), because of which the equation for $\hat{D}$ becomes linear with variable coefficients, which leads to "nonlinear" effects. For definiteness let $m>n>q$. We seek a solution of the form

$$\hat{D}=\hat{R}+\hat{Q}^{\pm}e^{\mp i\omega_1 t}+\hat{S}^{\pm}e^{\mp i\omega_2 t}+\hat{T}^{\pm}e^{\mp i(\omega_2-\omega_1)t},$$

in which we retain only the resonance matrix elements

$$R_{mm},\ R_{nn},\ R_{qq},\ Q^{+}_{mn},\ Q^{-}_{nm},\ S^{+}_{mq},\ S^{-}_{qm},\ T^{+}_{nq},\ T^{-}_{qn}.$$

The resonance part of the average dipole moment of the system is (omitting the indices of degeneracy)

$$\mathbf{d}(t) = 2\,\mathrm{Re}\,\{\mathbf{d}_{nm} Q^{+}_{mn} e^{-i\omega_1 t} + \mathbf{d}_{qm} S^{+}_{mq} e^{-i\omega_2 t} + \mathbf{d}_{qn} T^{+}_{nq} e^{-i(\omega_2-\omega_1)t}\}. \tag{2}$$

In the limiting case of a strong field Ψ and of a weak Φ

$$Q^{+}_{mn} = \Phi^{+}_{mn}\left\{\frac{A+Bi}{\Delta\omega_1 - \Delta\Omega_1 - i/\tau} + \frac{C+Di}{\Delta\omega_1 - \Delta\Omega_2 - i/\tau}\right\}, \tag{3}$$

where

$$\Delta\omega_1 = \omega_{mn} - \omega_1,\quad \Delta\omega_2 = \omega_{mq} - \omega_2,\quad \Delta\Omega_{1,2} = \Delta\omega_2/2 \mp \sqrt{(\Delta\omega_2/2)^2 + |\Psi|^2},$$

$$A = (\gamma_1\Delta\omega_2 - \gamma_2\Delta\Omega_1)/(\Delta\Omega_2 - \Delta\Omega_1),\quad A + C = \gamma_2,\quad B = (\gamma_3 - \gamma_2)/\tau(\Delta\Omega_2 - \Delta\Omega_1),\quad B + D = 0.$$

$$\gamma_k = \gamma_{mn} - k\gamma_{mq}\frac{|\Psi|^2}{\Delta\omega_2^2 + \tau^{-2} + 4|\Psi|^2},$$

and where $k = 1, 2, 3$; $|\Psi|^2 \equiv |\Psi_{mq}|^2$, and

$$\gamma_{mn} = (\omega_1/\omega_{mn})(\rho^0_m - \rho^0_n),\quad \gamma_{mq} = (\omega_2/\omega_{mq})(\rho^0_m - \rho^0_q).$$

From formula (3) it is evident that in general the absorption curve has two maxima.

Let $|\Psi|^2 \gg \Delta\omega_2^2,\ \tau^{-2}$. Then the positions of the two absorption maxima are given by the relations

$$(\Delta\omega_1)_1 = -|\Psi| + \Delta\omega_2/2 - \gamma_{mq}/8\gamma_2\tau^2|\Psi|,\quad (\Delta\omega_1)_2 = |\Psi| + \Delta\omega_2/2 + \gamma_{mq}/8\gamma_2\tau^2|\Psi|, \tag{4}$$

and the values of $I = Q^{+}_{mn}/\Phi^{+}_{mn}$ at these points are given by

$$I_1 = \tau\left\{\frac{\gamma_2}{2} + \frac{\Delta\omega_2}{4|\Psi|}\gamma_{mn}\right\},\quad I_2 = \tau\left\{\frac{\gamma_2}{2} - \frac{\Delta\omega_2}{4|\Psi|}\gamma_{mn}\right\}. \tag{5}$$

As is evident from (5), in this case the absorption curve has two approximately equal maxima, almost symmetrical relative to $\Delta\omega_1 = 0$.

If $|\Psi|^2,\ \tau^{-2} \ll \Delta\omega_2^2$, the positions of the maxima are

$$(\Delta\omega_1)_1 = -|\Psi|^2/\Delta\omega_2,\quad (\Delta\omega_1)_2 = \Delta\omega_2,$$

and their magnitude is given by

$$I_1 = \tau\left\{\gamma_{mn} - \frac{|\Psi|^2}{\Delta\omega_2^2}(\gamma_{mn} + \gamma_{mq})\right\},\quad I_2 = \tau\frac{|\Psi|^2}{\Delta\omega_2^2}(\gamma_{mn} - \gamma_{mq}).$$

In general the real part of the polarizability vanishes at three points. For $|\Psi|^2 \gg \Delta\omega_2^2,\ \tau^{-2}$, the positions of these points are

$$\Delta\omega_1 = \begin{cases} -|\Psi| + {}^1/_2\Delta\omega_2 + (2\gamma_3 - \gamma_2)/2\gamma_2\tau^2|\Psi| \\ \Delta\omega_2(\gamma_1/\gamma_2) \\ |\Psi| + {}^1/_2\Delta\omega_2 - (2\gamma_3 - \gamma_2)/2\gamma_2\tau^2|\Psi|. \end{cases}$$

The formulas obtained allow us to draw the following conclusions about the molecular generator and amplifier with auxiliary radiation:

1. If the auxiliary radiation is large, then the amplifier can operate at two frequencies.
2. The molecular generator is excited at those frequencies for which the real part of the polarizability is equal or close to zero. Therefore, for the type of generator considered, oscillations can arise at three frequencies, the oscillations being unstable for the central frequency.
3. The frequency of the oscillations of the generator will depend on both the frequency and amplitude of the auxiliary field.

Note added in proof (November 16, 1957). We note that the solution of (1) for both fields of arbitrary intensity is very cumbersome, but does not lead to results new in principle. Because of the transitions through the common level, which is an intermediate one, the absorption maximum is split for the frequency ω_1 as well as for ω_2.

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DETERMINATION OF PHASES OF MATRIX ELEMENTS OF THE S-MATRIX

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For reactions involving two or three channels and reactions with the formation of three low-energy particles, it is shown how the phases of the matrix elements of the S-matrix can be expressed in terms of measurable quantities by using the unitary property of the S-matrix and the invariance of the theory with respect to time reversal. The reactions $\tau^+ \rightarrow 2\pi^+ + \pi^-$ and $p + p \rightarrow D + \pi^+$ are considered.

In papers by Aizu, Fermi, and Watson[1,2] a relation has been established between the phases of the amplitudes for photoproduction of π mesons on a nucleon and the scattering phase shifts for scattering of π mesons by a nucleon; this relation follows from the unitary property of the S-matrix and the invariance of the theory with respect to time reversal. Subsequently analogous relations have been obtained in connection with a number of other reactions. But in all these cases the interaction in one of the channels was assumed to be weak. It seems that it may be of use to consider similar relations in the case of strong interactions.

The consequences of the unitary property of the S-matrix and of the invariance of the theory with respect to time reversal are usually discussed by means of the so-called K-matrix. That is, S is written in the form

$$S = (1 - iK)/(1 + iK)$$

and it is shown that in virtue of the properties of the S-matrix the matrix K is real and symmetric in a representation in which spin components are not included in the set of quantum numbers that specify the states of the system. If there are n channels, this means that the matrix elements of the S-matrix are expressed in terms of $n(n+1)/2$ independent real numbers. But the matrix elements of the matrix K are not directly measurable quantities. The quantities directly measured are the squares of the absolute values of matrix elements of the matrix $T = -i(S - 1)$.

The matrix elements of T, however, are complex. Therefore it is of interest to express the phases of the matrix elements of the T-matrix in terms of the squares of their absolute values. In a number of cases this can be done. At the same time, there appear several inequalities connecting the squares of the absolute values of the matrix elements of the T-matrix. The use of the K-matrix is particularly inconvenient for the consideration of states in which there are more than two particles. In these cases it is more convenient to start directly from the conditions for the unitary property of the S-matrix.

In the present paper these conditions are used to determine the phases of elements of the T-matrix in terms of scattering phase shifts and transition probabilities for the case of two or three channels. It is shown that the phases can also be determined in those cases, in which one of the channels contains three particles with small energy of relative motion. The imaginary part of the amplitude for the decay of a τ meson into three π mesons is expressed in terms of the real part and the scattering amplitudes for collisions between π mesons. At the same time it is shown that the interaction in the final state changes considerably the angular distribution of the decay products. The corrections caused by the interaction in the final state are determined for the phases of the matrix elements that fix the angular distribution and polarization in the reaction $p + p \rightarrow D + \pi^+$.

1. CASE OF TWO OR THREE CHANNELS

From the equation $SS^+ = 1$ it follows that

$$i(T - T^+) = -TT^+.$$

Let us consider the matrix elements of the right and left members between states $|a>$ and $|b>$ of the type indicated. Using the symmetry of S and the rule of matrix multiplication we get

$$\operatorname{Im}\langle a|T|b\rangle = \frac{1}{2}\sum_c \langle a|T|c\rangle\langle b|T|c\rangle^*, \tag{1}$$

$$\langle a|T|b\rangle = \langle a|T'|b\rangle\,\delta(\mathbf{P}_a - \mathbf{P}_b)\,\delta(E_a - E_b), \tag{2}$$

E_a, $\mathbf{P}_a$ are the energy and momentum of the system in the state $|a>$, and $<a|T'|b>$ is the T matrix on the energy surface. Substituting Eq. (2) into Eq. (1), we get the well known relation

$$\operatorname{Im}\langle a|T'|b\rangle = \frac{1}{2}\sum_c \langle a|T'|c\rangle\langle b|T'|c\rangle^*\,\delta(E_a - E_c)\,\delta(\mathbf{P}_a - \mathbf{P}_c). \tag{3}$$

Carrying out the summation with respect to $\mathbf{P}_c$, E_c on the right side of Eq. (3) we get

$$\operatorname{Im}\langle a|T'|b\rangle = \frac{1}{2}\sum_c \langle a|T'|c\rangle\langle b|T'|c\rangle\,\Gamma_c. \tag{4}$$

Here the summation extends only over states for which

$$\mathbf{P}_c = \mathbf{P}_a = \mathbf{P}_b; \quad E_c = E_a = E_b.$$

Γ_c is the phase volume of the states $|c>$ with energy E_c. Multiplying the right and left members by $\frac{1}{2}(\Gamma_a\Gamma_b)^{\frac{1}{2}}$ and writing

$$\frac{1}{2}\langle a|T'|b\rangle\sqrt{\Gamma_a\Gamma_b} = f_{ab}, \tag{5}$$

we get

$$\operatorname{Im} f_{ab} = \sum_c f_{ac} f_{bc}^*.$$

In the case $a = b$

$$\operatorname{Im} f_{ab} = \sum_c |f_{ac}|^2$$

and we get the usual result

$$f_{aa} = (\eta_a e^{2i\delta_a} - 1)/2i, \quad \eta_a = 1 - 4\sum_{c \neq a} |f_{ac}|^2. \tag{7}$$

Let us examine the case in which there are only two channels, as for example in the reaction $\pi^0 + p \to \pi^+ + n$ or in $K + N \to \Sigma + \pi$ if the initial state has isotopic spin $T = 0$ and we can neglect the processes $K + N \to \Sigma + \pi + \bar{n}$, $\Lambda + \bar{n} + \pi$ because of threshold effects for small energies of the K particle. In these cases we have

$$\operatorname{Im} f_{ab} = f_{ab} f_{ab} + f_{ab} f^{*}_{bb}. \tag{8}$$

Substituting Eq. (7) into Eq. (8), we get

$$f^{*}_{ab} \eta_a e^{2i\delta_a} - f_{ab} \eta_b e^{-2i\delta_b} = 0. \tag{9}$$

Since $\eta_a = \eta_b$ by Eq. (7), it follows from Eq. (9) that

$$f_{ab} = X_{ab} e^{i(\delta_a + \delta_b)}, \tag{10}$$

where X_{ab} is real. Equation (10) is a generalization of the relations of Aizu, Fermi, and Watson in the case of strong interactions.

In the case in which there are three channels, as for example in the scattering of K mesons by nucleons in the state with isotopic spin 1, when besides the elastic scattering there are possible the reactions $K + N \to \Lambda + \pi$ and $K + N \to \Sigma + \pi$, we have instead of Eq. (8), if the reactions $K + N \to \Lambda + 2\pi$, $\Sigma + 2\pi$ can be neglected,

$$\operatorname{Im} f_{ab} = f_{aa} f^{*}_{ab} + f_{ab} f^{*}_{bb} + f_{ac} f^{*}_{bc}. \tag{11}$$

Setting $f_{ab} = X_{ab} e^{i\varphi_{ab}}$ and using Eq. (7) we get

$$\frac{1}{2i}\left[\eta_a \exp(2i\delta_b - i\varphi_{ab}) - \eta_b \exp(-2i\delta_a + i\varphi_{ab})\right] X_{ab} + X_{bc} X_{ca} \exp(i(\varphi_{bc} - \varphi_{ca})) = 0.$$

Equating the absolute values of the two terms we have

$$\cos 2(\delta_a + \delta_b - \varphi_{ab}) = \frac{X^2_{ab} - 4X^4_{ab} - 2(X^2_{ac} + X^2_{bc}) X^2_{ab} - 2X^2_{ac} X^2_{bc}}{\eta_a \eta_b X^2_{ab}}. \tag{12}$$

Equation (12) solves the problem as stated, but in order for it actually to determine the phase it is necessary that the right member be less than unity in absolute value. This condition gives

$$X_{ab} X_{bc} X_{ac} \geqslant X^2_{ab} X^2_{bc} + X^2_{ac} X^2_{bc} + X^2_{ab} X^2_{ac}. \tag{13}$$

Or, solving for X_{ab} for example, we get

$$1/(1 + \eta_c) \leqslant X_{ab} / 2 X_{ac} X_{cb} \leqslant 1/(1 - \eta_c). \tag{14}$$

2. DECAY OF THE τ MESON

For small energy of the relative motion in one of the channels, we can treat also more complicated cases, for example those in which states with three particles are possible. As an example consider the decay of the τ meson, on the assumption that the decay interaction is invariant with respect to time reversal and that the spin of the τ meson is zero. We shall treat the π mesons as nonrelativistic and regard their wavelengths as large in comparison with the radius of interaction. In this case Eq. (3) gives

$$\operatorname{Im} \langle \tau^+ | T' | \tau_1 \tau_2 \tau_3 \mathbf{p}_3 \mathbf{k}_{12} \rangle = \frac{1}{2} \sum_{\tau'_1 \tau'_2 \tau'_3} \int d^3p'_3 d^3k'_{12} \langle \tau^+ | T' | \tau'_1 \tau'_2 \tau'_3 \mathbf{p}'_3 \mathbf{k}'_{12} \rangle \langle \tau_1 \tau_2 \tau_3 \mathbf{p}_3 \mathbf{k}_{12} | T' | \tau'_1 \tau'_2 \tau'_3 \mathbf{p}'_3 \mathbf{k}'_{12} \rangle \, \delta(M - E_{\mathbf{p}'_3, \mathbf{k}'_{12}}). \tag{15}$$

Here τ_1, τ_2, τ_3, are isotopic spin components for the three mesons, $\mathbf{k}_{12}$ is the momentum of the relative motion of the identical π mesons, $\mathbf{p}_3$ is the momentum of the third π meson in the rest system of the τ meson, and M is the mass of the τ meson.

Equation (15) contains the matrix element of the scattering of the three π mesons by each other, which can be represented as the sum of four contributions in the following way:

$$\langle\tau_1\tau_2\tau_3\mathbf{p}_3\mathbf{k}_{12}|T'|\tau_1'\tau_2'\tau_3'\mathbf{p}_3'\mathbf{k}_{12}'\rangle=\delta(\mathbf{p}_3-\mathbf{p}_3')\,\delta_{\tau_3\tau_3'}\langle\tau_1\tau_2\mathbf{k}_{12}|T'|\tau_1'\tau_2'\mathbf{k}_{12}'\rangle+\delta(\mathbf{p}_2-\mathbf{p}_2')\,\delta_{\tau_2\tau_2'}\langle\tau_1\tau_3\mathbf{k}_{13}'|T'|\tau_1'\tau_3'\mathbf{k}_{13}'\rangle$$
$$+\delta(\mathbf{p}_1-\mathbf{p}_1')\,\delta_{\tau_1\tau_1'}\langle\tau_2\tau_3\mathbf{k}_{23}|T'|\tau_2'\tau_3'\mathbf{k}_{23}'\rangle+\langle\tau_1\tau_2\tau_3\mathbf{p}_3\mathbf{k}_{12}|T'|\tau_1'\tau_2'\tau_3'\mathbf{p}_3''\mathbf{k}_{12}'\rangle_c. \tag{16}$$

In terms of Feynman diagrams the first three terms in Eq. (16) correspond to diagrams in which the line of one of the mesons is not connected with the others, and the last term corresponds to the connected diagrams. If we assume that the interaction between the π mesons has a definite radius, then the first three terms correspond to processes in which one of the particles was outside the region of interaction during the scattering of the other two. It is also clear that the first three terms contain scattering amplitudes of pairs of π mesons. Such a decomposition has meaning, since it will be shown that for small energy of the relative motion of the π mesons the contribution of the last term in Eq. (15) is small. In fact, substituting Eq. (16) into Eq. (15) we get

$$\operatorname{Im}\langle\tau^+|T'|\tau_1\tau_2\tau_3\mathbf{p}_3\mathbf{k}_{12}\rangle=\frac{1}{4}\mu k_{12}\sum_{\tau_1'\tau_2'}\int d\Omega_{\mathbf{k}_{12}'}\langle\tau^+|T'|\tau_1'\tau_2'\tau_3\mathbf{p}_3\mathbf{k}_{12}'\rangle\langle\tau_1\tau_2\mathbf{k}_{12}|T'|\tau_1'\tau_2'\mathbf{k}_{12}'\rangle^*$$
$$+\frac{1}{4}\mu k_{13}\sum_{\tau_1'\tau_3'}\int d\Omega_{\mathbf{k}_{13}'}\langle\tau^+|T'|\tau_1'\tau_2\tau_3'\mathbf{p}_2\mathbf{k}_{13}'\rangle\langle\tau_1\tau_3\mathbf{k}_{13}|T'|\tau_1'\tau_2'\mathbf{k}_{13}'\rangle^*+\frac{1}{4}\mu\mathbf{k}_{23}\sum_{\tau_2'\tau_3'}\int d\Omega_{\mathbf{k}_{23}'}\langle\tau^+|T'|\tau_1\tau_2'\tau_3'\mathbf{p}_1\mathbf{k}_{23}'\rangle$$
$$\times\langle\tau_2\tau_3\mathbf{k}_{23}|T'|\tau_2'\tau_3'\mathbf{k}_{23}'\rangle^*+\frac{\mu}{4}\sum_{\tau_1'\tau_2'\tau_3'}\int d^3p_3'\sqrt{\mu\left(M-\frac{3}{4}\frac{p_3'^2}{\mu}\right)}\,d\Omega_{\mathbf{k}_{12}'}$$
$$\times\langle\tau^+|T'|\tau_1'\tau_2'\tau_3'\mathbf{p}_3'\mathbf{k}_{12}'\rangle\langle\tau_1\tau_2\tau_3\mathbf{p}_3\mathbf{k}_{12}|T'|\tau_1'\tau_2'\tau_3'\mathbf{p}_3'\mathbf{k}_{12}'\rangle^*_c. \tag{17}$$

Here μ is the mass of the π meson.

If we consider the right member as a function of the energy of the relative motion of the π mesons, then it can be seen that (because of the tendency of all the matrix elements to a constant limit) the first three terms in Eq. (17) go to zero like k_{12}, k_{13}, or k_{23}, i.e., proportional to $(M-3\mu)^{\frac{1}{2}}$, while the last term varies as $(M-3\mu)^2$. If we neglect all powers of k_{12}, k_{13}, k_{23} except the first, we get

$$\operatorname{Im}\langle\tau^+|T'|\tau_1\tau_2\tau_3\mathbf{p}_3\mathbf{k}_{12}\rangle=\pi\mu\sum_{\tau_1'\tau_2'\tau_3'}\langle\tau^+|T'|\tau_1'\tau_2'\tau_3'\rangle\left\{\delta_{\tau_3\tau_3'}a_{\tau_1\tau_2}^{\tau_1'\tau_2'}k_{12}+\delta_{\tau_2\tau_2'}a_{\tau_1\tau_3}^{\tau_1'\tau_3'}k_{13}+\delta_{\tau_1\tau_1'}a_{\tau_2\tau_3}^{\tau_2'\tau_3'}k_{23}\right\}, \tag{18}$$

Here $<\tau^+|T'|\tau_1'\ \tau_2'\ \tau_3'>$ is the amplitude for decay with neglect of the energy of the π mesons (a real quantity) and $a_{\tau_1\tau_2}^{\tau_1'\tau_3'}$ is the scattering amplitude for collisions between π mesons of zero energy. If we make use of invariance in isotopic spin space, then we have

$$a_{\tau_1\tau_2}^{\tau_1'\tau_2'}=\sum_{T=0,2}(1\,1\,\tau_1\tau_2|1\,1\,TT')(1\,1\,\tau_1'\tau_2'|1\,1\,TT')\,a_T, \tag{19}$$

Here $(1\tau_1 1\tau_2|11TT')$ are Clebsch-Gordan coefficients, and a_0, a_2 are the scattering amplitudes of π mesons in the states with isotopic spins $T = 0, 2$.

It must be noted that Eq. (18) is valid only under the condition that the scattering of the π meson by the π meson does not have resonance character, with $ka \sim 1$ already at very small energies, since in this case the neglect of high powers of k is not permissible. Equation (18) is of interest in the following connection: if we expand the decay amplitude in terms of spherical harmonics, then for example for the decay $\tau^+ \to 2\pi^+ + \pi^-$ we have

$$\langle\tau^+|T'|1,1,-1;\mathbf{p}_3\mathbf{k}_{12}\rangle=\sum_L P_L(\cos\vartheta)\langle\tau^+|T'|1,1-1Lp_3,Lk_{12}\rangle. \tag{20}$$

Here L is the angular momentum of the relative motion of the two π mesons; $L = 0, 2, 4$, since particles 1 and 2 are identical; L is also equal to the angular momentum of the third particle relative to the center of mass of the first two, because the spin of the τ meson is taken to be zero; and ϑ is the angle between p_3 and k_{12}. If we neglect the interaction of the π mesons in the final state, then for small p_3 and

k_{12} the matrix element in Eq. (20) is proportional to $p_3^L k_{12}^L$, i.e., the coefficient of $\cos^2\vartheta$ in the angular distribution is about $(M - 3\mu)^2$. But according to Eq. (18) the coefficient of $\cos^2\vartheta$ is different from zero already in the first order in $k \sim (M - 3\mu)^{\frac{1}{2}}$, since

$$k_{13} = \sqrt{\frac{9}{16} p_3^2 + \frac{1}{4} k_{12}^3 + \frac{3}{4} p_3 k_{12} \cos\vartheta},$$

and k_{23} has an analogous dependence on the angles.

Accordingly the interaction of the π mesons in the final state decidedly changes the energy dependence of the matrix elements that determine the angular distribution. In another paper it will be shown how this fact makes it possible to determine the scattering amplitudes for collisions between π mesons from the angular distribution of the decay products. In a similar way one can treat for example the photoproduction of two π mesons from a nucleon for energies near the threshold.

3. THE REACTION $p + p \rightarrow D + \pi^+$

As another example of the application of Eq. (3) we can take the reaction

$$p + p \rightarrow D + \pi^+.$$

The amplitude for the reaction in the center-of-mass system can be written in the form

$$\langle \mathbf{k} s_z | T' | \sigma_1\sigma_2 \mathbf{p}\rangle = \sum (l\, 1 m s_z | l\, 1 j\mu)(l's'm's'_z | l's'j\mu)(^1/_2\, ^1/_2\, \sigma_1\sigma_2 | ^1/_2\, ^1/_2\, s's'_z)\langle jlk | T' | jl's'p\rangle Y_{lm}(\Omega_{\mathbf{k}}) Y_{l'm'}(\Omega_{\mathbf{p}_s}). \tag{21}$$

Here $\mathbf{k}$ is the momentum of the meson and $\mathbf{p}$ that of the proton. The remaining notations are obvious.

Gell-Mann and Watson have shown[3] that if one neglects the scattering of the π meson by the deuteron the phase of the quantity $\langle j\ell k | T' | j\ell' s' p'\rangle$ is equal to the scattering phase shift for proton-proton scattering in the state $| j\ell' s' p'\rangle$. Equation (3) enables us to calculate the correction to the phase caused by the interaction in the final state. Since we are interested in the reaction amplitude for small energy, we confine ourselves to $\ell = 0, 1$. Under these conditions ℓ and ℓ' are good quantum numbers and it follows from Eq. (3) that:

$$\begin{aligned} \mathrm{Im}\langle jlk | T' | jl's'p\rangle = {} & \frac{1}{2} pM \langle jlk | T' | jl's'p\rangle \langle jl's'p | T' | jl's'p\rangle^* + \frac{1}{2} k\mu \langle jlk | T' | jlk\rangle \langle jl's'p | T' | jlk\rangle^* \\ & + \sum_{L\Lambda s''} \int q_1^2\, dq_1 p_1^2\, dp_1 \delta(E - E_{p_1, q_1}) \langle jlk | T' | jLp_1\Lambda q_1 s''\rangle_0 \langle jl's'p | T' | jLp_1\Lambda q_1 s''\rangle_0^* \\ & + \sum_{L\Lambda s''} \int q_1^2\, dq_1 p_1^2\, dp_1 \delta(E - E_{p_1, q_1}) \langle jlk | T' | jLp_1\Lambda q_1 s''\rangle_+ \langle jl's'p | T' | jLp_1\Lambda q_1 s''\rangle_+^*. \end{aligned} \tag{22}$$

M and μ are the masses of the proton and the π meson. The third and fourth terms arise from the respective processes

$$p + p \rightarrow p + p + \pi^0 \rightarrow D + \pi^+, \quad p + p \rightarrow p + n + \pi^+ \rightarrow D + \pi^+,$$

which are possible if the energy of the π^+ is larger than the binding energy of the deuteron. In the case in question these terms need not be small in comparison with the second, since at low energy the nucleons have a resonance interaction.

Setting

$$\langle jlk | T' | jl's'p\rangle = \rho e^{i\varphi};\ \frac{1}{2} pM \langle jl's'p | T' | jl's'p\rangle = (e^{2i\delta_{l'}}\eta_{l'} - 1)/2i; \quad \frac{1}{2} k\mu \langle jlk | T' | jlk\rangle = (e^{2i\delta_l}\eta_l - 1)/2i;\ \varphi = \delta_{l'} + \varphi'$$

and using equations analogous to Eq. (22) for the amplitudes appearing in the right member, we get the first order correction

$$\begin{aligned} \varphi' = \delta_l + \frac{1}{\rho} \sum_{L\Lambda s''} \int q_1^2\, dq_1 p_1^2\, dp_1 \delta(E - E_{p_1, q_1}) \,\mathrm{Re}\, \langle jlk | T' | jLp_1\Lambda q_1 s''\rangle_0, \\ \times |\langle jl's'p | T' | jLp_1\Lambda q_1 s''\rangle_0| + \frac{1}{\rho} \sum_{L\Lambda s''} \int q_1^2\, dq_1 p_1^2\, dp_1 \delta(E - E_{p_1, q_1}) \,\mathrm{Re}\langle jlk | T' | jLp_1\Lambda q_1 s''\rangle_+ |\langle jl's'p | T' | jLp_1\Lambda q_1 s''\rangle_+|. \end{aligned}$$

Since in this approximation

$$\operatorname{Im}\langle jlk\,|\,T'\,|\,jLp_1\Lambda q_1 s''\rangle = \rho\,\frac{1}{2}\,pM\,|\,\langle jl's'p\,|\,T'\,|\,jLp_1\Lambda q_1 s''\rangle\,|\,,$$

the right member contains experimentally measurable quantities.

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THE FEYNMAN PATH INTEGRAL FOR THE DIRAC EQUATION

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It is shown that, subject to certain assumptions about the motion of the electron, the Feynman path integral is identical with the propagation function of the Dirac equation.

1. INTRODUCTION

The problem considered below was initiated by Feynman and has been discussed in Refs. 1—4. The main result of the first paper[1] is the relation

$$R(x_1t_1\,|\,x_2t_2) = \int e^{iS\{x(t)\}/h}\delta x(t). \tag{1}$$

Here $R(x_1t_1|x_2t_2)$ is the solution of the Schrödinger equation which becomes $\delta(x_1 - x_2)$ for $t_1 = t_2$;

$$S = \int_{t_1}^{t_2} L\{\dot{x}(t), x(t)\}\,dt$$

is the increase of the classical action along a path; and $\int \delta x(t)$ denotes integration (summation) over all trajectories $x(t)$ for which $x(t_1) = x_1$ and $x(t_2) = x_2$.

Together with certain other rules, the relation (1) gives a closed formulation of quantum mechanics. This formulation is mathematically equivalent to the usual formalism, but the Feynman approach has a number of advantages. In particular, the formal solution of any problem is obtained in the form of an infinitely multiple integral over the paths. There is a hope that just this approach will give a simple and perspicuous system of concepts and notations which will make it possible to get beyond the framework of perturbation theory in quantum mesodynamics.

Another virtue of the method is the space-time description and the absence of operators. One can follow in thought the motion of the particle—the motion turns out to be a special case of a Markov random process.

One further important advantage is the connection of the method with the Hamilton principle of least action. The formula (1) gives so to speak the probability of those virtual trajectories which are considered in the formulation of the principle of least action. And whereas the classical equation of motion is obtained firstly from a Lagrangian (specific information about the system) and secondly from the principle of least action, the quantum equation of motion in the nonrelativistic domain (the Schrödinger equation) is obtained firstly from the same classical Lagrangian and secondly from the Feynman principle, Eq. (1). The extension of this scheme to the relativistic domain is the purpose of the present paper. In analogy with the use of Hamilton's principle in classical physics we can expect that the principle (1) will be important for the solution of various theoretical problems.

But there is another side of the question: Strictly speaking, Eq. (1) is not applicable to any physical system except the electromagnetic field and the nonrelativeistic case, which is of little interest. Indeed, if we substitute in $S\{x(t)\}$ in Eq. (1) the classical relativistic Lagrangian, we do not get on the left-hand side the solution of the Dirac equation, above all because of the absence of the spinor indices in the classical picture.

But it turns out to be possible, even in the classical picture, to give a geometrical meaning to the spinor indices (see below) which makes it possible to obtain the Dirac equation from the Feynman principle. The considerations on which the proposed solution is based are explained in the conclusion. Use is also made of some ideas from Refs. 3—6.

2. A TWO-DIMENSIONAL ANALOGUE OF THE DIRAC EQUATION

In considering the problem of the relativistic formulation of the Feynman principle we use as an example the equation

$$[(\partial/\partial t+iA_t)+\sigma_1(\partial/\partial r-iA_r)+i\mu\sigma_3]\psi=0,\quad \psi=\begin{matrix}\psi_1(r,t)\\ \psi_{-1}(r,t)\end{matrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{2}$$

In this case the classical action has the form

$$\int(-\mu\sqrt{dt^2-dr^2}+A_r\,dr-A_t\,dt). \tag{3}$$

This is, as it were, half of the Dirac equation, and the results of the analysis will remain in force for the electron. All of the developments for Eq. (2) are simple, and the paths can be drawn on paper.

For what follows it is convenient to take instead of Eq. (2) the more general equation

$$[(\partial/\partial t+iA_t)+\sigma_1(\partial/\partial r-iA_r)+\sigma_3(\partial/\partial s+i\mu)]\psi(r,t,s)=0. \tag{4}$$

If the coefficients and the initial condition do not depend on s, then Eq. (4) reduces to Eq. (2).

We give the name of the propagation function $R(r, t, s\,|\,r', t', s')$ to the solution of the equation which reduces to the δ function at the initial time. This function has the property

$$\psi(r,t,s)=\int R(r,t,s\,|\,r',t',s')\,\psi(r',t',s')\,dr'\,ds'. \tag{5}$$

In order to construct the integral (1) for Eq. (4) we must first relate the propagation function of Eq. (4) to the classical action for small time intervals. Noting the results of Refs. 3—6, we shall seek this relation in a mixed representation, i.e., we shall number the columns of the propagation matrix with eigenvalues of the matrix σ_3 (which we denote by n, with $n = \pm 1$) and the rows with eigenvalues of the matrix σ_1 (which we denote by m, with $m = \pm 1$).

To avoid difficulties with continuous paths we shall assume that the particle can exist only at the nodal points of a space-time lattice with the period ϵ; then Eq. (4) is replaced by the corresponding equation in finite differences. The relationship expressing the Feynman principle for this equation has the form

$$R_{nm}(r+\varepsilon_r,\,t+\varepsilon,\,s+\varepsilon_s\,|\,r,\,t,\,s)=\eta_{nm}\exp\{-i\mu\varepsilon_s+iA_r\varepsilon-iA_t\varepsilon\}. \tag{6}$$

Here R_{nm} is the propagation function, $\epsilon_r = m\epsilon$ is the change of the coordinate, $\epsilon_s = n\epsilon$ is the change of the proper time, and

$$\eta_{nm}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}_{nm}=\pm\frac{1}{\sqrt{2}}.$$

Substituting Eq. (6) into Eq. (5) we get, for $\epsilon \to 0$, Eq. (4).

The paths of the particle are obtained from Eq. (6) in the following way. Suppose that at the time $t = 0$ the particle was at the point $r = 0$, $s = 0$ with the spinor index m, and that at the next instant of time, $t = \epsilon$, the spinor index is equal to n. From Eqs. (5) and (6) we get that at the instant $t = \epsilon$ the wave function is different from zero only at the point $r = m\epsilon$, $s = n\epsilon$. To the four possible combinations of initial and final indices there correspond the four paths shown in Fig. 1.

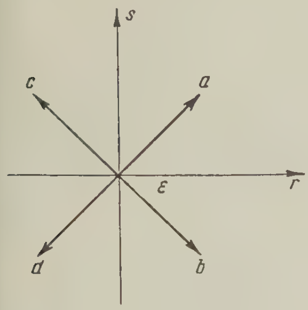

FIG. 1. (a) $n = 1$, $m = 1$; (b) $n = -1$, $m = 1$; (c) $n = 1$, $m = -1$; (d) $n = -1$, $m = -1$

Let us follow the path of the particle (Fig. 2). If $n = 1$ at the instant $t = 0$, the transitions a and c are possible, and at the instant $t = \epsilon$ the particle will be either at A or at B. If the particle has come to the point A, the transition has been of the type c and the particle has the spinor index $m = -1$. For $m = -1$ the possible transitions are c and d (Fig. 1), and consequently at the instant $t = 2\epsilon$ the particle will be either at C or at D. If the particle has come to the point D, then $n = -1$ and the possible transitions are b and d, and so on.

To each path in the space r, s there corresponds a definite succession of indices (and vice versa), and thus the geometrical significance of the spinor indices becomes clear. Namely, the two values of the number n denote two modes of intersection of the world line with the r, t plane, and $|\psi_1(r, t)|^2$ and $|\psi_{-1}(r, t)|^2$ are the probabilities for finding the particle moving through the point (r, t) in the directions of larger and smaller s.

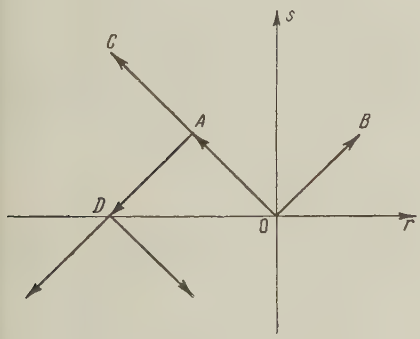

FIG. 2

This interpretation of necessity leads to the appearance of spinor indices on the propagation function. The transition $r', t' \to r, t$ occurs along one of the paths, and the prescription of the path means the prescription of the displacements along the path. Therefore the transition amplitude is obtained by addition of the amplitudes for the separate paths, and the amplitude for each path is obtained by multiplying together the amplitudes (6) of the displacements. If spinor indices are prescribed at the instants t' and t, for example n' and n, the transition is possible only along a particular type of path, Fig. 3, and only these paths contribute to the transition amplitude. The four amplitudes that form the matrix propagation function correspond to the four possibilities for the paths (see Fig. 3).

These considerations, i.e., the repeated application of Eq. (6), give for $t - t' = N\epsilon$

$$R_{nn'}(r, t, s \mid r', t', s') = \sum_{\epsilon_r^k,\, \epsilon_s^k} E_{nn'}\, e^{iS}, \tag{7}$$

with the condition $\sum \epsilon_r^k = r - r'$, $\sum \epsilon_s^k = s - s'$; $\epsilon_r^k = m_k\epsilon = \pm\epsilon$ is the change of the coordinate of the particle during the time interval (t_{k+1}, t_k), with $t_k = t' + k\epsilon$, and $\epsilon_s^k = n_k\epsilon = \pm\epsilon$ is the change of the proper time during this same interval. The summation is taken over all the successive displacements, i.e., over all the paths in the space, and

$$E_{nn'} = \eta_{\ m_1}\eta_{m_1n_2}\eta_{n_2m_2}\eta_{m_2n_3}\cdots\eta_{n_Nm_N}\eta_{m_Nn'} = \pm 2^{-N}.$$

Unlike the case of Eq. (1), part of the paths occur with the opposite sign. But all the paths for which the sign of ϵ_s, i.e., the sign of the change of the proper time, does not change along the path, occur with the sign +. Only these paths remain in the nonrelativistic approximation

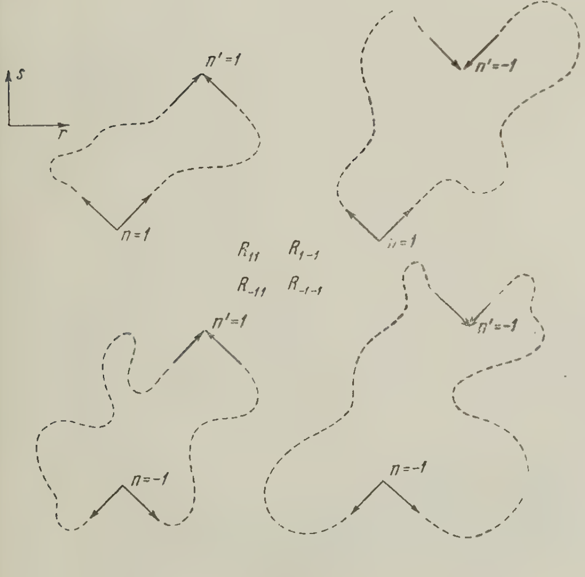

FIG. 3

$$S = -\sum \mu \varepsilon_s^k + \sum A_r(t_k, r_k)\varepsilon_r^k - \sum A_t(t_k, r_k)\varepsilon = -\sum \mu(s_k - s_{k-1}) + \sum A_r(t_k, r_k)(r_k - r_{k-1}) - \sum A_t(t_k, r_k)(t - t_{k-1}),$$

$$S \to -\int \mu ds + \int A_r dr - \int A_t dt \quad \text{for } \varepsilon \to 0.$$

The propagation function $R(r, t|r', t')$ is the sum over paths with arbitrary change of the proper time. Therefore

$$R_{nn'}(r, t|r', t') = \int_{-\infty}^{\infty} R_{nn'}(r, t, s|r', t', s')\,d(s - s') = \sum_{\varepsilon_r^k \varepsilon_s^k} E_{nn'} e^{iS}, \quad \sum \varepsilon_r^k = r - r'. \tag{8}$$

The meaning of the factor $E_{nn'}$ is partly revealed if we go over to a new representation $\psi'_n = e^{i\pi n/4}\psi_n$ and introduce the notation $\delta n_k = n_{k+1} - n_k$; we get

$$R'_{nn'} = e^{i\pi(n-n')/4} R_{nn'} = \sum \exp\left\{iS + \frac{i\pi}{4}\sum m_k \delta n_k\right\}, \quad n_1 = n, \quad n_{N+1} = n'. \tag{9}$$

From this it can be seen that the sign factor $E_{nn'}$ is the exponential of the sum along the path. In the classical action there appears an additional term depending on the signs of the displacement. This additional term can be shown to be equal to the quantity $(\pi h/2)(M - L)$, where M is the number of turns to the right and L the number to the left along the path; that is:

$$R'_{nn'} = \sum \exp\left\{iS + \frac{i\pi}{2}(M - L)\right\}. \tag{9'}$$

Here one needs to take account only of turns with change of n or only those with change of m ($\sum m_k \delta n_k = -\sum n_k \delta m_k$). In the latter case reflection in the line $S = \text{const}$ of a part of the path bounded by that line does not change the action along the path. But on reflection left and right turns are interchanged, so that if the reflected part of the path contains a turn the new path occurs in the sum with the opposite sign, and the contributions of the two paths cancel. There remain only transitions that are asymmetrical with respect to the turns, and the particle turns out to be "rotating."

Here there is an analogy with the expression for the nonrelativistic propagation function for several electrons, $R = \sum \exp(iS + i\pi P)$, where P is the number of interchanges of the final coordinates. In the relativistic theory the antisymmetry with respect to interchanges is supplemented by an antisymmetry with respect to replacement of one of the left turns by a right turn.

Let us consider further the calculation of the sums (7) and (8). If there is no external field, i.e., $A_r = A_t = 0$, these sums are easily calculated by the method of Markov (the motion of the particle is a Markov process). In Eq. (8) we must replace the condition $\sum \epsilon_r^k = r - r'$ by the factor

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left\{i\left(\sum \varepsilon_r^k - r - r'\right)p'/\varepsilon\right\}dp'.$$

The expression under the sign of summation is the product of N two-rowed matrices. Let the transformation Q bring each of the matrices to diagonal form with eigenvalues λ_1 and λ_2. Then

$$R = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} Q^{-1} \exp\{-i(r - r')p'/\varepsilon\}\,dp', \quad \lambda_{1,2} = 1 \pm i\varepsilon\sqrt{\mu^2 + p^2}, \quad p = p'/\varepsilon.$$

For $\epsilon \to 0$, $N\epsilon = t$

$$R \to \frac{1}{2\pi}\int_{-\infty}^{\infty} Q \begin{pmatrix} \exp\{it\sqrt{\mu^2 + p^2}\} & 0 \\ 0 & \exp\{-it\sqrt{\mu^2 + p^2}\} \end{pmatrix} Q^{-1} \exp\{-i(r - r')p\}\,dp.$$

3. TRANSITION TO CLASSICAL MECHANICS

The transition to small h is made as in Ref. 1. In this connection, there appear in the relativistic domain new features that are of interest from the point of view of the relation between the classical and quantum pictures of the motion.

Let us go back to Eq. (7) and divide the time $t - t'$ into q intervals of size a, such that the external field can be regarded as constant for regions of space or time that are smaller than a (the speed of light is equal to 1). Let r', $r_1, \ldots, r_q = r$ and s', $s_1, \ldots, s_q = s$ be the coordinates and proper times of the particle at the instants t', $t' + a, \ldots, t' + qa = t$. One can divide the summation over the paths into two stages and first take the sum over all paths that pass through r', $r_1, \ldots, r$ and s', $s_1, \ldots, s$ and then the sum over all values of r_1 and s_1. Carrying out the first summation and setting $\epsilon = 0$, we get

$$R(r, t, s \mid r', t', s') = \int \exp\left\{i \sum (-\mu \Delta s_k + A_r(t_k, r_k)\Delta r_k - A_t(t_k, r_k)\Delta t_k)\right\} \prod_k \rho(\Delta s_k, \Delta r_k, \Delta t_k)\, dr_k ds_k,$$

$$\Delta s_k = s_k - s_{k-1}, \quad \Delta r_k = r_k - r_{k-1}, \quad \Delta t_k = t_k - t_{k-1}, \tag{10}$$

ρ will obviously be the propagation function of a free particle with $\mu = 0$.

For $h \to 0$ (i.e., $\mu \to \infty$), because of the rapid oscillation of the factor $\exp\{-i\mu(sk - s_{k-1})\}$, the contributions from regions with smooth variation of ρ cancel each other and there remain only paths along which ρ has a singularity, i.e., paths with $\Delta s_k^2 = \Delta t_k^2 - \Delta r_k^2$. In this approximation

$$R(r, t \mid r', t') = \int \exp\left\{i \sum \left(\pm \mu \sqrt{\Delta t_k^2 - \Delta r_k^2} + A_r \Delta r_k - A_t \Delta t_k\right)\right\} \prod_k \rho'(\Delta r_k, \Delta t_k)\, dr_k. \tag{11}$$

Here the largest contribution will be from the region with slow variation of the phase of the exponential function (we denote this phase by S), i.e., with $\partial S/\partial r_k = 0$. If the sign in front of the square root changes on the passage of the particle through r_k, then, as can be shown, there will be no stationary solution (if the field varies smoothly). There remain only paths for which the sign of the root does not change and $\partial S/\partial r_k = 0$, i.e., the principle of least action is satisfied.

Thus for $h \to 0$ the picture of the motion undergoes an essential change. Thus for the quantization of the relativistic equations, i.e., formally, for the reconstruction of the theory from its limiting case, the nonrelativistic Feynman principle or, what is the same thing, the replacement of the momenta by operators, turns out to be insufficient.

4. DERIVATION OF THE NONRELATIVISTIC FEYNMAN PRINCIPLE

The nonrelativistic approximation means that all distances are large in comparison with the Compton wavelength h/mc. Therefore, as in Sec. 3, $\mu \Delta s_k \to \infty$ and Eq. (10) goes over into Eq. (11). In the nonrelativistic approximation all the elements of the matrix ρ' in Eq. (11) go to zero except ρ_{11} if the plus sign is taken in front of the root, or ρ_{-1-1} if the minus sign is taken. Therefore, as in Sec. 3, there remain only paths for which the sign in front of the root does not change. We get the nonrelativistic Feynman principle (1) for the classical Lagrangian (3).

5. THE FEYNMAN PRINCIPLE FOR THE DIRAC EQUATION

We take the Dirac equation in the Majorana representation:

$$[\gamma_\nu (\partial / \partial x_\nu - i A_\nu) + \mu]\psi = 0; \qquad \gamma_1 = -\sigma_3, \quad \gamma_2 = \sigma_1, \quad \gamma_3 = \sigma_3\sigma_1\rho_3\rho_1, \quad \gamma_4 = i\sigma_3\sigma_1\rho_1;$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In addition to the usual representation, in which the matrices σ_3 and ρ_3 are diagonal (in this representation we shall denote the spinor index by the letter α), we take a representation in which ρ_1 and σ_1 are diagonal (and in which we denote the spinor index by β). The Feynman relation for an infinitely small time interval is obtained in this mixed representation by a simple generalization of Eq. (6) and has the form

$$R_{\alpha\beta}(x_\nu + \varepsilon_\nu | x_\nu) = \pm \frac{1}{2} \exp\{i(-\mu\varepsilon_s + A_\nu \varepsilon_\nu)\}, \qquad \varepsilon_\nu = \pm \varepsilon, \quad i\varepsilon_s = \pm \varepsilon, \tag{12}$$

Here ϵ is the change of the proper time.

The choise of signs is determined by the values of the indices α, β, i.e., the Dirac equation is replaced by the relation

$$\psi_\alpha(x_\nu) = \sum_\beta \eta_{\alpha\beta} \exp\{i(-\mu\varepsilon_s + A_\nu\varepsilon_\nu)\}\psi_\beta(x_\nu - \varepsilon_\nu), \quad \varepsilon_\nu = f(\alpha, \beta)\varepsilon, \quad f(\alpha, \beta) = \pm 1, \quad \eta_{\alpha\beta} = \pm {}^1/_2.$$

The four values of the initial spinor indices and the four values of the final spinor indices give 16 possible combinations, and each combination prescribes a choice of the signs, i.e., a path of the electron from the center of the four-dimensional cube of side 2ϵ to one of the 16 vertices (space x, y, z, t if ϵ_s is prescribed, or x, y, z, s if ϵ_t is prescribed).

For finite time intervals $t - t' = N\epsilon$,

$$R(x_\nu, s \,|\, x'_\nu, s) = 4^{-N} \sum_{\varepsilon_\nu^k, \varepsilon_s^k} \pm e^{iS}, \qquad \sum \varepsilon_\nu^k = x_\nu - x'_\nu, \quad \sum \varepsilon_s^k = s - s', \quad S \to -\int \mu ds + \int A_\nu dx_\nu, \quad \varepsilon \to 0.$$

The sign of the path is the product of the signs of the successive displacements. As in the two-dimensional case, this sign is determined by the numbers of left and right turns (in the planes x, y and z, s). The expression (12) is interesting for the reason that not only does the path, i.e., the succession of coordinate values, determine the succession of spinor indices, but also conversely the succession of indices determines the path in space. One can renounce the latter requirement and seek the Feynman relation in the form

$$R_{\alpha\alpha'}(x_\nu + \varepsilon_\nu, x_\nu) = \sum_\gamma \eta(\alpha, \alpha', \gamma) \exp\{i(-\mu\varepsilon_s + A_\nu\varepsilon_\nu)\}, \quad \varepsilon_\nu = f(\alpha, \alpha', \gamma)\varepsilon. \tag{13}$$

The sum over γ denotes the sum over all displacements with prescribed α and α'. For each choice of the elementary displacements (i.e., each concrete meaning of γ) and each representation one can choose $f(\alpha, \alpha', \gamma)$ (i.e., the geometrical meaning of the indices) and the sign factor $\eta(\alpha, \alpha', \gamma)$, using the requirement that Eq. (13) go over into the Dirac equation for $\epsilon \to 0$.

6. CONCLUSION

Let us examine the physical and mathematical considerations which have made it possible to generalize the Feynman principle.

The first of these is a more precise interpretation of the meaning of relativistic wave functions. As has been shown, the relativistic wave function $\psi_\alpha(x_\nu)$ gives not only the probability of the particle being at the point x_ν, but also certain indications about the motion of the particle through this point. This information is contained in the index α. All the paths that arrive at x_ν are divided into four classes, and $\psi_\alpha(x_\nu)$ means the sum over all paths of the type α.

Although the physical facts are simple, it is hard to write them down in the language of continuous paths. Therefore one has to regard space and time as discrete and correspondingly to alter the meaning of the symbol $\delta x(t)$ as compared with the use in Ref. 1 [cf. Eq. (1)]. The formulas become somewhat cumbersome but, as can be seen from Sec. 2, the infinitely multiple sums encountered here can be calculated relatively simply.

In Sec. 3 one more difference from Feynman's results appears. The domain of paths of integration (summation) in Eq. (7) turns out to be broader than the domain of comparison paths in the classical principle of least action. The form of the action is just the same: $-\int \mu\, ds + \int A_\nu dx_\nu$ (or $\int A_\alpha dx_\alpha$, $\alpha = 1, 2, 3, 4, 5$ in a five-dimensional notation). But in the principle of least action one considers only paths for which $ds^2 = dt^2 - dx^2$ and $ds > 0$. In Eq. (7) one does not have this limitation and the "actual" classical picture (i.e., the picture for which the Feynman principle gives the Dirac equation) turns out to be symmetrical in relation to all five coordinates.

Finally there remains the antisymmetry of left and right turns considered in Sec. 2. This antisymmetry is obtained automatically if we replace the usual requirement on the Lagrangian—that it leave the principle of least action invariant—by the requirement of covariance of the Feynman principle.[1]

In conclusion I warmly thank V. Ia. Fainberg for direction and advice.

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NONRADIATIVE RECOMBINATION OF ELECTRONS AT IMPURITY CENTERS IN N-TYPE GERMANIUM

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Nonradiative ("phonon") recombination of a current carrier at a Coulomb impurity center in n-type germanium is investigated at helium temperatures. Interaction between an electron and acoustic vibrations of the lattice is taken into account by successive diagonalization of the original Hamiltonian by a unitary transformation. The tensor nature of the effective carrier mass and also the general expression for the deformation potential are taken into account in the calculations.

1. FORMULATION OF THE PROBLEM

The complex character of the energy bands in germanium and silicon, which has been revealed in cyclotron ("diamagnetic") resonance experiments and in certain other experiments, again raises the problem of calculating the energy levels and wave functions of the electron in an impurity center in these crystals. For n-type samples there exist experimental data concerning the existence of a "longitudinal" effective mass, μ_ℓ, and a "transverse" effective mass, μ_t, of the electron. The tensor character of the effective mass has made it necessary to reexamine the old "scalar" model of impurity centers.[1,2] In these investigations, however, the interaction between the electron and the lattice vibration has not been taken into account. On the other hand, Deigen's[3] work has shown, that when one takes account of such interactions (deformation potential) for impurities in homopolar crystals, there appear in some cases comparatively deep levels, located below the bottom of the conduction band. Apparently, with respect to this so called "condenson" effect,[4] one encounters in Ge a case of weak binding. General expressions for the deformation potential in n-type germanium were obtained by Dumke,[5] who calculated, in the same paper, the scattering of current carriers in an ideal crystal by interaction with the lattice vibrations.

The purpose of the present work is to take into account the deformation potential for an electron in an impurity center by means of the weak-coupling approximation, and also to construct a corresponding theory for non-radiative recombination. It must be emphasized, that within the framework of the proposed method of calculation, the discrete and continuous spectra of electrons in impurity centers are considered together and thus the well known difficulties in this question[6] can be avoided.

2. ENERGY LEVELS AND QUANTUM STATES OF THE SYSTEM

It is known, that the conduction band in n-type germanium has eight degenerate energy minima or energy valleys, oriented along the [111] direction of the first Brillioiun zone. In the vicinity of each of these minima, the constant energy surfaces have the form of elongated ellipsoids, whose major axis is oriented along the [111] direction. Lampert[1] has shown, following the analysis of Koster and Slater, that the ground state of the electron in an impurity center can be found by the method of effective masses, by considering only the wave function belonging to a given energy valley. Such a treatment remains valid when one sets out to construct a theory of thermal transitions, if the thermal dissociation of the impurity center gives rise to electrons with a wave vector k, which does not exceed the bounds of the given valley. For the inverse process of thermal recombination, this essentially presupposes that the probability for capture of an electron with a vector k, associated with a particular valley, by a bound state associated with another energy valley, is small. Such "intervalley" capture can occur in principle, but will not be considered in this paper.

In what follows, it is convenient to choose a coordinate system such that the Z-axis coincides with the major axis of the ellipsoid associated with the given valley. If one takes into account the interaction of the electron with the potential due to the strain and shear deformations,[5] then the generalized Hamiltonian of the system, after transformation to the second quantization representation, can be written in the form

$$\hat{H} = \frac{\hat{p}_X^2 + \hat{p}_Y^2}{2\mu_t} + \frac{\hat{p}_Z^2}{2\mu_l} + V(R) + \sum_{j=1}^{3} \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}j} \hat{B}_{\mathbf{q}j}^{+} \hat{B}_{\mathbf{q}j} + \hat{H}_i. \tag{1}$$

Here

$$V(R) = -ze^2/\varepsilon R \tag{2}$$

is the Coulomb energy of the electron; ϵ is the dielectric constant; $\hat{B}_{qj}^{+}$ and $\hat{B}_{qj}$ are Bose operators, and H_i is the operator for the interaction of the electron with the free phonon field:

$$\hat{H}_i = G_1 \sum_{j=1}^{3} \sum_{\mathbf{q}} (\hbar\omega_{qj}/2C_{qj}L^3)^{1/2} f_j(\theta) (\hat{B}_{\mathbf{q}j} e^{i\mathbf{q}\mathbf{R}} + e^{-i\mathbf{q}\mathbf{R}} \hat{B}_{\mathbf{q}j}^{+}), \quad f_1(\theta) = 1 + (G_2/G_1)(3\cos^2\theta - 1),$$

$$f_2(\theta) = (3G_2/G_1)\cos\theta\sin\theta\cos\chi, \quad f_3(\theta) = (3G_2/G_1)\cos\theta\sin\theta\sin\chi. \tag{3}$$

Summation over j in Eq. (1) and (3) includes one longitudinal and two transverse waves for every acoustic-wave vector[1] **q**. In Eqs. (3), θ is the angle between **q** and the Z axis for the given energy valley; χ is the angle between the direction of polarization $\mathbf{e}_{q_2}$ of the acoustic wave and the plane passing through the Z axis and the normal to **q**; the energy constants G_1 and G_2 are chosen to give the best fit between the theoretical calculations of the mobility of current carriers in germanium and experimental measurements.[5] The elastic constants G_{qj}, as shown in Ref. (5), can be considered independent of the wave vector **q** and for germanium crystals one can substitute certain average values corresponding to longitudinal and transverse waves; L^3 is the volume of the fundamental cell of the crystal.

In what follows, we shall change for convenience from the variables X, Y, and Z in Eqs. (1) and (3) to the variables x, y, and z defined thus:

$$x = X; \quad y = Y; \quad z = Z/\gamma; \quad \gamma = \sqrt{\mu_t/\mu_l}. \tag{4}$$

We take advantage of the method of successive diagonalizations of the Hamiltonian by unitary transformations[8] in the form proposed by Bohm and Pines and employed subsequently by Toyazawa[9] in his investigation of the "electron polaron." Such a method allows us to take into account the interaction of the electron with the phonon field with an accuracy to second-order terms.

The transformed Hamiltonian Eq. (1) has the form

$$\hat{\Lambda} = \hat{U}^{-1}\hat{H}U. \tag{5}$$

The unitary operator $\hat{U}$ is chosen to have the form

$$\hat{U} = \exp(i\hat{S}/\hbar), \tag{6}$$

where the Hermitian operator $\hat{S}$ is defined below.

Omitting the details of the calculations, we arrive at

$$\hat{H} = \hat{p}^2/2\mu_t + V(\sqrt{x^2+y^2+\gamma^2z^2}) + \sum_{q,j} \hbar\omega_{qj}\hat{b}_{qj}^+\hat{b}_{qj} + \hat{H}' + \hat{H}''. \tag{7}$$

Here

$$\hat{b}_{qj} = e^{i\hat{S}/\hbar}B_{qj}e^{-i\hat{S}/\hbar};\ \hat{b}_{qj}^+ = e^{-i\hat{S}/\hbar}\hat{B}_{qj}^+e^{i\hat{S}/\hbar}, \tag{8}$$

$\hat{H}'$ and $\hat{H}''$ are second-order operators.

$$\hat{H} = -\frac{i}{2\hbar}[\hat{S},\ \hat{\Lambda}_i],\ \hat{H}'' = -\frac{i}{\hbar}[\hat{S}, V(\sqrt{x^2+y^2+\gamma^2z^2})], \tag{9}$$

$$\hat{\Lambda}_l = e^{i\hat{S}/\hbar}\hat{H}_l e^{-i\hat{S}/\hbar} = G_1\sum_{q,j}\sqrt{\frac{\hbar\omega_{qj}}{2C_{qj}L^3}}f_l(\theta)[\hat{b}_{qj}\exp\{i(xq_x+yq_y+\gamma zq_z)\} + \exp\{-i(xq_x+yq_y+\gamma zq_z)\}\hat{b}_{qj}^+]. \tag{10}$$

The operator $\hat{S}$ is defined by equating the sum of first order terms to zero, thus;

$$\hat{\Lambda}_i - \frac{i}{\hbar}\left[\hat{S}, \frac{\hat{p}^2}{2\mu_t}\right] - \frac{i}{\hbar}\left[\hat{S}, \left(\sum_{q,j}\hbar\omega_{qj}\hat{b}_{qj}^+\hat{b}_{qj}\right)\right] = 0. \tag{11}$$

It can be put into a more convenient form as follows:

$$\hat{S} = \sum_{q,j}(\hat{\alpha}_{qj}\hat{b}_{qj}\exp\{i(xq_x+yq_y+\gamma zq_z)\} - \exp\{-i(xq_x+yq_y+\gamma zq_z)\}\hat{\alpha}_{qj}^+\hat{b}_{qj}^+). \tag{12}$$

After evaluating the commutators in Eq. (11), we arrive at:

$$\hat{S} = -i\hbar G_1\sum_{q,j}\sqrt{\frac{\hbar\omega_{qj}}{2C_{qj}L^3}}f_j(\theta)(\hat{b}_{qj}\hat{L}^{-1}e^{i(xq_x+yq_y+\gamma zq_z)} - \hat{b}_{qj}^+e^{-i(xq_x+yq_y+\gamma zq_z)}\hat{L}^{-1}), \tag{13}$$

$$\hat{L} = \frac{\hbar^2}{2\mu_t}(q_x^2+q_y^2+\gamma^2q_z^2) - \frac{\hbar}{\mu_t}(q_x\hat{p}_x + q_y\hat{p}_y + \gamma q_z\hat{p}_z) + \hbar\omega_{qj}. \tag{14}$$

Using the given value of $\hat{S}$, it is possible to arrive at explicit expressions for $\hat{H}'$ and $\hat{H}''$.

The calculations are greatly simplified if the operator L^{-1} is expanded into a series in the ratio of the electron momentum to the phonon momentum, which is a small quantity. Such an expansion can be made for all q, which satisfy the relation:

$$q \gg p/\hbar \approx \sqrt{\mu_t E_{\text{ion}}/\hbar^2} \approx 10^6\ \text{cm}^{-1}. \tag{15}$$

Since $q_{\max} = (6\pi^2/a^3)^{\frac{1}{3}} \approx 10^8\text{cm}^{-1}$ (a is the lattice constant), the inequality (15) embraces, as will be evident from Eq. (20), the region of the largest most important values of (q).

Calculation of $\hat{H}''$ yields:

$$\hat{H}'' = -\frac{ie^2\hbar^2G_1}{\mu_t\varepsilon}\sum_{q,j}\sqrt{\frac{\hbar\omega_{qj}}{2C_{qj}L^3}}f_j(\theta)\frac{xq_x+yq_y+\gamma^3zq_z}{(x^2+y^2+\gamma^2z^2)^{3/2}}\ \frac{\hat{b}_{qj}\exp\{i(xq_x+yq_y+\gamma zq_z)\} - \hat{b}_{qj}^+\exp\{-i(xq_x+yq_y+\gamma zq_z)\}}{[(\hbar^2/2\mu_t)(q_x^2+q_y^2+\gamma^2q_z^2)+\hbar\omega_{qj}]^2}. \tag{16}$$

To determine $\hat{H}'$, note the following. The eigenvalues of the energy operator $\hat{H}$ can be obtained if Eq. (7) is first averaged over oscillations of the field. Then, in the energy of the electron there appear additional terms only for the averaged operator $\hat{H}'$. They have the form

$$\langle\hat{H}'\rangle = \mathscr{E}_0 - (\hat{p}_x^2+\hat{p}_y^2)/2M_t - \hat{p}_z^2/2M_l. \tag{17}$$

Here $<\ldots>$ signifies an averaging over the virtual phonon field. To simplify the calculation, we consider the case of low temperatures, when we can with great accuracy say that

$$\langle\hat{b}_{qj}^+\hat{b}_{qj}\rangle = O(L^{-3});\ \langle\hat{b}_{qj}\hat{b}_{qj}^+\rangle = 1 + O(L^{-3}). \tag{18}$$

The expression for $\mathcal{E}_0$ has the form

$$\mathcal{E}_0 = -\frac{\hbar G_1^2}{2L^3}\sum_{\mathbf{q},j} f_j^2(\theta)\,\omega_{qj}/C_{qj}[\cdots], \quad [\cdots] = [(\hbar^2/2\mu_t)(q_x^2+q_y^2+\gamma^2 q_z^2)+\hbar\omega_{qj}]. \tag{19}$$

The quantities $1/M_t$ and $1/M_\ell$ are respectively given by

$$1/M_t = (G_1^2\hbar^3/\mu_t^2 L^3)\sum_{\mathbf{q},j} f_j^2(\theta)\,\omega_{\mathbf{q}j}\,qq_z^2/C_{qj}[\cdots]^3 = (G_1^2\hbar^3/\mu_t^2 L^3)\sum_{\mathbf{q},j} f_j^2(\theta)\,\omega_{\mathbf{q}j}\,qq_y^2/C_{qj}[\cdots]^3,$$

$$1/M_l = (\gamma^2 G_1^2\hbar^3/\mu_t^2 L^3)\sum_{\mathbf{q},j} f_j^2(\theta)\,\omega_{\mathbf{q}j}\,qq_z^2/C_{qj}[\cdots]^3. \tag{20}$$

Thus, one can speak of an equivalent Hamiltonian $\widetilde{H}$, in which $\hat{H}'$ is included:

$$\widetilde{H} = \frac{\hat{p}_x^2+\hat{p}_y^2}{2\mu'_{\mathrm{eff}}} + \frac{\hat{p}_z^2}{2\mu''_{\mathrm{eff}}} + V(\sqrt{x^2+y^2+\gamma^2 z^2}) + \sum_{\mathbf{q},j}\hbar\omega_{qj}\,\hat{b}_{qj}^{+} b_{qj} + \mathcal{E}_0 + \hat{H}''$$

$$1/\mu'_{\mathrm{eff}} = 1/\mu_t - 1/M_t; \quad 1/\mu''_{\mathrm{eff}} = 1/\mu_t - 1/M_l. \tag{21}$$

The Hamiltonian $\widetilde{H}$ describes a motion of a condenson, similar to that of a quasi-particle in the Coulomb field of the impurity center.

In the approximation of an isotropic crystal, μ_{eff} and $\mathcal{E}_0$ are readily determined. If one assumes $\gamma = 1$; $G_2 = 0$, $\omega_q = w_\ell q$ (w_ℓ is the "longitudinal" velocity of sound), then changing the summation in Eqs. (19) and (20) into an integration in spherical coordinates, we find:

$$\mathcal{E}_0 = -\frac{1}{4\pi^2}\frac{\hbar G_1^2}{\rho w_l}\int_0^{q_{\max}}\frac{q^2 dq}{\hbar w_l + \hbar^2 q/2\mu}, \quad \mu_{\mathrm{eff}} = \mu\left[1+\frac{4\mu^2 G_1^2}{3\pi^2\rho w_l\hbar^3}\int_0^{q_{\max}}\frac{q^2 dq}{(q+2\mu w_l/\hbar)^3}\right], \tag{22}$$

where ρ is the density of the crystal. This result agrees exactly with the calculation (by Pekar and Krivoglaz) of the ground-state energy and the effective mass of the condeson in a homopolar crystal; they used a different method.[10]

If we take into account the tensor character of the effective mass and neglect for simplicity the shear deformation in describing the deformation potential, we obtain the following result:

$$\mathcal{E}_0 = -\frac{w_l G_1^2\mu_t^3}{\pi^2\rho\hbar^3\sqrt{1-\gamma^2}}\int_0^{\hbar q_{\max}/2\mu_t w_l}\frac{x^2}{\sqrt{x(x+1)}}\ln\left|\frac{\sqrt{x(1-\gamma^2)}+\sqrt{x+1}}{\sqrt{x(1-\gamma^2)}-\sqrt{x+1}}\right|dx, \tag{23}$$

$$\frac{\mu'_{\mathrm{eff}}}{\mu_t} \approx 1+\frac{\mu_t}{M_t} = 1+\frac{\mu_t^2 G_1^2}{4\pi^2\rho w_l\hbar^3\sqrt{1-\gamma^2}}\int_0^{\hbar q_{\max}/2\mu_t w_l}\frac{x\,dx}{(1+x)^2}\left(3+\frac{1+x+|\gamma^2-1|x}{\sqrt{(1-\gamma^2)(1+x)x}}\ln\frac{\sqrt{x(1-\gamma^2)}-\sqrt{1+x}}{\sqrt{x(1-\gamma^2)}+\sqrt{1+x}}\right),$$

$$\mu''_{\mathrm{eff}}/\mu_t \approx 1+\mu_t/M_l; \quad M_l \approx M_t/100\gamma^2. \tag{24}$$

Including the shear deformation in Eqs. (19) and (20) complicates the mathematical description of the final results somewhat, but does not change their numerical value very much.

The fact, that in the approximation adopted in this analysis, the binding energy of the condenson $\mathcal{E}_0$ does not depend on the quantum number of the electronic state in the Coulomb field of the impurity, becomes understandable if one notes that the radius of the electronic state is very large even for the ground state (the ionization energy of the impurity center is about 0.01 ev).

In what follows, the operator $\hat{H}''$ plays the role of a small perturbation that induces quantum transitions between stationary states of the system, obtained from solutions of Schrödinger's equation

$$(\widetilde{H} - \hat{H}'')\Psi = E\Psi. \tag{25}$$

The wave function in the zeroth approximation assumes the form.

$$\Psi_{s\ldots n_{qj}\ldots} = \psi_s(x,y,z)\prod_{\mathbf{q},j}\Delta(n_{qj}-n_{qj}^0) \tag{26}$$

where $\psi_S(x, y, z)$ is defined by the equation

$$\frac{p_x^2 + p_y^2}{2\mu'_{\text{eff}}}\psi_s + \frac{p_z^2}{2\mu''_{\text{eff}}}\psi_s - \frac{ze^2}{\varepsilon\sqrt{x^2 + y^2 + \gamma^2 z^2}}\psi_s = E_s\psi_s. \tag{27}$$

The second term in Eq. (26) is the wave function of the oscillators of the field in the second-quantization representation (Δ is the Kronecker function).

The ground state can be obtained from Eq. (27) by the variational method. Let us neglect the differences in the effective masses of the "longitudinal" condensons and adopt the approximation:

$$\mu'_{\text{eff}} \approx \mu''_{\text{eff}} \approx \mu_t, \tag{28}$$

We can then take advantage, for example, of Lampert's results.[1] We must take it into account here that the energy level of an electron in an impurity center is always shifted by an amount $\mathscr{E}_0$, which equals for Ge crystals amounts to 8 – 10% of the theoretically calculated ionization energy of the center.

3. NONRADIATIVE RECOMBINATION OF A CONDENSON AT A COULOMB CENTER

The problem of non-radiative recombination in Ge crystals has been treated in Refs. (11) through (13). These investigations are not satisfactory from our viewpoint, because of the way that the interaction with the lattice vibrations is handled. Thus some of these authors completely neglect the lattice vibrations in the zeroth approximation; others consider it in the tight-coupling approximation. The use of the adiabatic method in Ge cannot really be justified, in view of the exceedingly small value of the thermal-dissociation energy of the impurity. Besides, the calculations mentioned above are based on the "scalar" model of the center, which therefore reduces the theoretical significance of the results.

Let us consider one of the problems of the theory of nonradiative transitions, namely, the thermal transition of a condenson from the ground state of the Coulomb center to the conduction band under the action of the perturbation operator $\hat{H}''$. Furthermore, let us assume that the condition described by Eq. (28) is satisfied, thus allowing us, in effect, to change the word "condenson" to "electron." The uniqueness of the diagonalization of the Hamiltonian manifests itself above all in the theory of thermal transitions, principally because the form of the perturbation operator is changed substantially. Let us choose for simplicity the following initial form for the electron wave function ψ_S

$$\psi_s = A\exp\{-\alpha^2(x^2 + y^2) - \beta^2 z^2\}. \tag{29}$$

Here α^2 and β^2 are variable parameters, determined by the condition that the energy E in Eq. (27) is a minimum.

The wave function of the final state of the electron in a crystal with a large dielectric constant is usually assumed to have the form of a plane wave.[11,13] However, as has been correctly pointed out by Gummel and Lax, the replacement of the Coulomb function by the continuous spectrum of a plane wave leads to a lowering of the transition probability by one or two orders of magnitude.

Solutions of Eq. (27) for electronic states in the continuous spectrum cannot be obtained directly. One can hardly expect success from the generalized variational method for this case. On the other hand, in crystals like Ge, the large effective-mass anisotropy allows one to introduce the following approximation into the potential-energy terms of the Hamiltonian for large distances from the center:

$$\gamma^2 = \mu_t/\mu_l \approx 0. \tag{30}$$

Transforming Eq. (27) into cylindrical coordinates and making use of Eq. (30), one can arrive at an approximate expression for the wave function of the continuous spectrum

$$\psi_{k_0,\, m,\, k_z} = C_{k_0} e^{ik_z z} e^{im\varphi} (2ik_0\rho)^m e^{-ik_0\rho} F\left(m + \frac{i\alpha_0}{k} + \frac{1}{2};\ 2m + 1;\ 2ik_0\rho\right),$$

$$\rho = \sqrt{x^2 + y^2};\ k_0 = \sqrt{k_x^2 + k_y^2};\ \alpha_0 = \mu e^2/\varepsilon\hbar^2, \tag{31}$$

F (a, b, c) is the confluent hypergeometric function and C_{k_0} is a normalization constant.

Calculation of the matrix element from such a function is an extremely complicated mathematical problem, whose solution can be obtained only by numerical methods. To illustrate some of the characteristics of the theory of thermal transitions which follow from the theory described here, we should try to obtain the final equations for the transition probabilities by analytic methods. In view of this, let us limit ourselves to considering functions of the form

$$\psi_{k_x, k_y, k_z} = L^{-3/2} \exp\{i(k_x x + k_y y + k_z z)\}. \tag{32}$$

The transition probability of the system from the state $(s \ldots n_{qj} \ldots)$ to all states $(k_x, k_y, k_z \ldots n'_{qj} \ldots)$ should be averaged over all initial vibrational states of the lattice and summed over all final states of the electron and the lattice. Utilizing the method of summing probabilities given by Lax,[14] one can obtain the following expression for the transition probability:

$$W_{\text{ion}} = \frac{1}{\hbar^2} \sum_{\mathbf{k}} \sum_{\mathbf{q}, j} \int_{-\infty}^{+\infty} |(s|H''|k)|^2 [(\bar{n}_{qj} + 1) e^{i\omega_{qj} t} + \bar{n}_{qj} e^{-i\omega_{qj} t}] \exp\left\{\frac{i}{\hbar}(\mathscr{E}_k - E_s) t\right\} dt. \tag{33}$$

Here $\bar{n}_{qj}$ is the average value of the quantum number:

$$\bar{n}_{qj} = [\exp(\hbar\omega_{qj}/k_0 T) - 1]^{-1} \approx \exp(-\hbar\omega_{qj}/k_0 T) \tag{34}$$

and $\mathscr{E}_k$ is defined as

$$\mathscr{E}_k = \hbar^2 k^2 / 2\mu_t.$$

The matrix element that enters into Eq. (33) has the following form in cylindrical coordinates:

$$(s|H''|k) = \sqrt{\frac{AB_{qj}}{L^3}} \int_0^{\infty} e^{-\alpha^2\rho^2} \rho\, d\rho \int_{-\infty}^{+\infty} \frac{e^{-\beta^2 z^2 + i\gamma z(q_z - k_z/\gamma)}}{(\rho^2 + \gamma^2 z^2)^{3/2}}\, dz$$

$$\int_0^{2\pi} \exp\{i\rho[(q_x - k_x)\cos\varphi + (q_y - k_y)\sin\varphi]\} (\rho q_x \cos\varphi + \rho q_y \sin\varphi + \gamma^3 z q_z)\, d\varphi,$$

$$A = (2/\pi)^{3/2} \beta\alpha^2, \quad B_{qj} = \omega_{qj} e^4 \hbar^5 f_j^2(\theta) G_1^2 / 2L^3 C_{qj} \varepsilon^2 \mu_t^2 [\ldots]^4. \tag{35}$$

Let us change the summation over q in Eq. (33) to an integration in spherical coordinates, with the polar axis chosen to coincide with the original Z axis. To do this we must bear in mind that there are $(L/2\pi)^3 d\tau_q$ acoustic vibrations in an element of volume in q-space, associated with a given direction of polarization. Integration over the variable t yields a δ-function, with which one can immediately calculate the integral over q:

$$W_{\text{ion}} = \frac{L^3}{4\pi^2\hbar^2} \sum_{\mathbf{k}} \sum_{j=1}^{3} \frac{q_{0j}^2}{w_j} \int_0^{2\pi} d\varphi \int_0^{\pi} |(s|H''|k)|^2 \exp\left(-\frac{\hbar w_j q_{\theta j}}{k_0 T}\right) \sin\theta\, d\theta. \tag{36}$$

In the above the dispersion law we used $\omega_{qj} = w_j q_{0j}$, where q_{0j} is the wave number, determined from the conservation of energy:

$$\hbar w_j q_{0j} = \hbar^2 k^2 / 2\mu_t + |E_s|. \tag{37}$$

At low temperatures, the quantity q_{0j} that enters into the matrix element of Eq. (36) can be written

$$q_{0j} \approx |E_s| / \hbar w_j. \tag{38}$$

It is assumed that in crystals like Ge and Si the anisotropy does not alter the inequalities, which are valid at helium temperatures for isotropic crystals,[12] namely

$$\mathscr{E}_k \approx k_0 T \ll k\Theta_D - |E_s|, \tag{39}$$

where Θ_D is the Debye temperature.

The integrals in the matrix element of Eq. (35) can be evaluated by expanding the integrand in a series in terms of the small-valued parameter γ^2 ($\gamma^2_{\text{Ge}} = 0.059$, cf. Ref. 1). Furthermore, in accordance with Eq. (39), the quantities k_x, k_y, and k_z/γ can be neglected compared with the wave vector of the lattice, q_{0j}. Integration over φ in Eq. (35) leads to Bessel functions, and subsequent integrations proceed in a straightforward way. Finally, we obtain

$$(s\,|\,H''\,|\,k) = \frac{i\pi^2}{2\alpha\beta}\sqrt{\frac{ABq_{0j}}{L^3}}\,q_{0j}^2\sin^2\theta\exp\left(-\frac{q_{0j}^2\sin^2\theta}{4\alpha^2}\right)F\left(\frac{3}{2};\,2;\,\frac{q_{0j}^2\sin^2\theta}{4\alpha^2}\right)\left\{1+\frac{\gamma^2}{2\beta^2}\left[-\frac{q_{0j}^2\cos^2\theta}{2}+3\alpha^2\frac{F\,(^5/_2;\,2;\,q_{0j}^2\sin^2\theta/4\alpha^2)}{F\,(^3/_2;\,2;\,q_{0j}^2\sin^2\theta/4\alpha^2)}\right]\right\} \tag{40}$$

(θ is the angle between the vector q and the z axis, and F (a, b, c) is the confluent hypergeometric function). Further, substituting Eq. (40) into Eq. (36), we replace summation over q by integration in spherical coordintes, with the polar axis chosen as before to coincide with the original z axis. The integrand in the integral with respect to the variable $\cos\theta = x$ has a sharp maximum near the point $x_m = 1 - \gamma^2$; the functions that change very little with increasing x can be taken outside the integral and replaced by their values at $x = x_m$. Omitting the details of the calculation, we arrive at the result:

$$W_{\text{ion}} = \frac{16\alpha^2 G_1^2 e^4 \mu_t^{7/2}(k_0T)^{3/2}\exp\{-\,|\,E_s\,|\,/\,k_0T\}}{3\pi\beta\varepsilon^2\gamma^4\hbar^8}\sum_{j=1}^{3}\frac{f_j^2(x_m)}{C_j q_{0j}^3}\left[1-\frac{\gamma^2 q_{0j}^2}{2\beta^2}\left(1-12\gamma^2\Phi(\gamma)\right)\right], \tag{41}$$

$$\Phi(\gamma) = \int_0^1\frac{(1-x^2)^3}{[1-(1-\gamma^2)\,x^2]^4}\,dx \approx \frac{1+3\gamma^2}{4}\left[\frac{11}{3}(\gamma^2-1)+(2+\gamma^2)\ln\left|\frac{4-\gamma^2}{\gamma^2}\right|\right]. \tag{42}$$

The hypergeometric functions are replaced by their asymptotic values.[15]

Summation over j in Eq. (41) yields,

$$W_{\text{ion}} = \frac{16\alpha^2 G_1^2 w_l^3 e^4 \mu_t^{7/2}(k_0T)^{3/2}\exp(-\,|\,E_s\,|/k_0T)}{3\pi\beta C_L\varepsilon^2\gamma^4\hbar^5 E_s^3}\left\{\left(1+\frac{2G_2}{G_1}\right)^2\left[1-\frac{\gamma^2E_s^2}{2\hbar^2\beta^2 w_l^2}(1-12\gamma^2\Phi)\right]+\frac{18\gamma^2 C_L G_2^2}{C_T G_1^2}\left(\frac{w_t}{w_l}\right)^3\left[1-\frac{\gamma^2E_s^2\,(1-12\gamma^2\Phi)}{2\hbar^2\beta^2 w_t^2}\right]\right\}.$$

Following Ref. 5, we use for the elastic constants C_L and C_T the arithmetic means of the elastic constants corresponding to the directions of propagation [110], [100] and [111] for longitudinal and transverse waves. Note that the expression that is subtracted from unity in the brackets of Eq. (41) is the second term in the expansion of the matrix element in terms of the small parameter γ^2. For Ge it amounts to only several percent of the first term in the series.

In a manner similar to that presented above one can calculate the probability for capture of a current carrier by the ground state of the impurity center.

The corresponding calculation leads to:

$$w_{\text{rec}} \approx \frac{32\sqrt{2\pi}\,\alpha^2\mu_t^2 e^4}{3\beta\varepsilon^2\hbar^2E_s^3\gamma^4L^3}\left[\frac{(G_1+2G_2)^2\,w_l^3}{C_L}+\frac{18\gamma^2G_2^2w_t^3}{C_T}\right] \tag{44}$$

The same result is obtained directly by applying the principle of detailed balancing.

One can evaluate α^2, β^2, and E_s in Eq. (44) with the help of the variational method, and obtain the following:

$$\beta^2 = \gamma\sqrt{2}\,\alpha^2;\quad \alpha^2 = \left(\frac{ze^2\mu_t}{2\varepsilon\hbar^2}\right)^2\frac{2\pi}{(1-\gamma/\sqrt{2})^3};\qquad E_s = \frac{\hbar^2}{2\mu_t}(\beta^2+2\alpha^2)-\frac{ze^2\sqrt{2\pi}\,\alpha\beta}{\varepsilon\,(\beta^2-\alpha^2\gamma^2)^{1/2}}\,. \tag{45}$$

Using the following values[16] for the parameters of Ge:

$\rho = 5.4$ g/cm^3; $\mu_t = 0.083$ m; $\gamma^2 = 0.059$; $\epsilon = 16$; $w_l = 5.16 \times 10^5$ cm/sec; $w_t = 3.2 \times 10^5$ cm/sec; $C_L = 14.2 \times 10^{11}$ dynes/cm^2; $C_T = 5.8 \times 10^{11}$ dynes/cm^2; $G_1 \approx -G_2$; $G_2 = 5.84$ ev.

We obtain:

$\alpha^2 = 2.62 \times 10^{12}\ \text{cm}^{-2}$; $\beta^2 = 0.905 \times 10^{12}\ \text{cm}^{-2}$; E = 0.0115 ev

Equation (44) yields

$$w_{\text{rec}} = 1.48\cdot 10^{-8}\,L^{-3}\ \text{sec}^{-1}. \tag{46}$$

Replacement of the plane wave by the more accurate function, Eq. (31), should increase the calculated transition probability by one or two orders of magnitude. For low temperatures this, apparently, is a general result.[9,12]

Note, that the value of W_{rec} calculated by Sclar and Burstein,[17] who used the equations of Gummel and Lax[12] and took into account the Coulomb function for the electron in the continuous spectrum, is $3.5 \times 10^{-9}\ L^{-3}\ \text{sec}^{-1}$. This is three orders of magnitude smaller than the value determined from data[17] on the photoconductivity of Ge and Si. It follows directly from Eq. (46) that, within the framework of the theory presented here, we have succeeded qualitatively in eliminating the above discrepancy with experiment. We have also indicated what must be done for an accurate quantitative calculation. The question of competing mechanisms of recombination[18] will be investigated separately.

In conclusion I wish to express my gratitude to Iu. E. Perlin, M. F. Deigen and V. A. Moskalenko for discussions of the results of this work.

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THE MAGNETIC STRUCTURE OF FLUORIDES OF THE TRANSITION METALS

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The method of Ref. 1 is applied to the problem of the magnetic structure of the antiferromagnetic crystals MnF_2, CoF_2, FeF_2, and NiF_2. It is shown that crystals with such symmetry may possess a weak ferromagnetism.

In Ref. 1 the problem of the weak ferromagnetism of certain antiferromagnetic substances was treated on the basis of Landau's theory of phase transitions of the second kind. In the present work the same method is used to study the magnetic structure of MnF_2, CoF_2, FeF_2, and NiF_2.

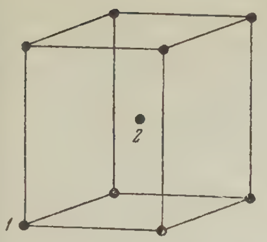


FIG. 1. Unit cell of MnF_2. Only the Mn^{++} ions are shown.

These crystals are isomorphic; they have a simple tetragonal lattice and the symmetry of space group D_{4h}^{14}.[2] In their unit cell there are two metallic ions in positions (000) and $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$ (cf. Fig. 1; here only the metallic ions are represented). The magnetic structure of the substances under consideration was studied neutronographically by Erickson.[3] It was found that they are all antiferromagnetic; the spins of the ions located at the centers of the crystal cells are equal in magnitude, and opposite in sign, to the spins of the ions located at the lattice points. In the case of MnF_2, CoF_2, and FeF_2, the spins are directed along the crystal axis. In the case of NiF_2, the small intensity of the magnetic scattering by the nickel ions makes it impossible to determine with great certainty the direction of the spins; Erickson himself concluded that the picture that agrees best with the observed intensities is one in which the spins make an angle of about 10° with the crystal axis.

It follows from Erickson's work that the antiferromagnetic transition, in all four fluorides, occurs without change of the unit cell; therefore their magnetic structure is described by the mean magnitudes $\mathbf{s}_1$ and $\mathbf{s}_2$ of the spins of the two metallic ions located in the unit cell. As in Ref. 1, we expand the thermodynamic potential Φ of the crystal as a power series in the components of $\mathbf{s}_1$ and $\mathbf{s}_2$. For this purpose it is convenient to introduce the vector $\mathbf{m} = \mathbf{s}_1 + \mathbf{s}_2$, representing the magnetic moment of the unit cell, and $\boldsymbol{\ell} = \mathbf{s}_1 - \mathbf{s}_2$. It is easily verified that in all transformations of the symmetry group D_{4h}^{14}, the vectors $\mathbf{m}$ and $\boldsymbol{\ell}$ transform independently of each other. With a view to consideration of the behavior of a crystal over a wide temperature range, rather than specifically in the neighborhood of the transition, we shall write the expansion of Φ in powers of the magnetic moment $\mathbf{m}$ and of the unit vector $\boldsymbol{\gamma}$ in the direction of $\boldsymbol{\ell}$.

The thermodynamic potential consistent with the crystal symmetry, to terms of the fourth order, has the form

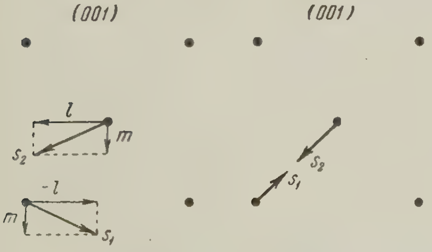
$$\Phi = \frac{a}{2}\gamma_z^2 + \frac{B}{2}\mathbf{m}^2 + \frac{b}{2}m_z^2 + e(\gamma_x m_y + \gamma_y m_x) + \frac{D}{2}(\mathbf{m}\gamma)^2 + 2d(\mathbf{m}\gamma)\gamma_x\gamma_y + \frac{f}{4}\gamma_z^4 + \frac{g}{2}\gamma_x^2\gamma_y^2.$$

The terms in $\mathbf{m}^2$ and $(\mathbf{m}\boldsymbol{\gamma})^2$ correspond to exchange interaction of the spins. The coefficients a, e, d stand in front of terms that represent relativistic interaction of the spins, and they are therefore proportional to the small quantity $(v/c)^2$, where v is the electron speed; the coefficients f and g are proportional to $(v/c)^4$. The presence of the term $\gamma_x m_y + \gamma_y m_x$ in the expansion of the thermodynamic potential leads, as in Ref. 1, to the result that the crystal in some states may possess a small spontaneous moment, of the order of $(v/c)^2$ times the nominal.

Crystals of the symmetry under consideration can exist in three magnetic states; one or another corresponds to a minimum of Φ, according to the relation among a', f, and g', where

$$a' = a + e^2/B,\quad g' = g + 2e^2D/B(B+D) + 2d^2/(B+D).$$

Namely, in the case in which $2a' + f < 0 < g'$ or $2a' + f < g' < 0$, the spins of the two metallic ions are equal and of opposite signs and are directed along the crystal axis (state I). Such an antiferromagnetic structure was observed by Erickson in MnF_2, CoF_2, and FeF_2. In the case $g' > 0$, $2a' + f > 0$ (state II_1), the spins of the two ions, in the first approximation, are compensated and are directed along the [100] (or [010]) axis of the tetragonal cell. In the next approximation, however, they are rotated toward one another about the crystal axis, remaining in the (001) plane, so that there results a spontaneous magnetic moment $m_0 = |e|/B(\sim(v/c)^2)$, directed along the [010] (or [100]) axis, i.e., perpendicular to the vector γ (Fig. 2a). Finally, in the case $g' < 0$, $2a' + f > g'$ (state II_2), the spins, in the first approximation, are likewise compensated, but are directed along one of the [110] axes. In the next approximation there arises a spontaneous moment $m_0 = |e + d|/(B + D)$, directed along the same [110] axis; that is, the spins of the metallic ions become unequal in magnitude without change of direction (Fig. 2b).



a b

FIG. 2. a—orientation of the spins in state II_1; b—orientation of the spins in state II_2.

All three antiferromagnetic states, I, II_1, and II_2, can originate directly by a phase transition of the second kind from the paramagnetic state. Transitions between them occur in the range where the coefficient a' is approaching zero. When a' becomes small $(\sim (v/c)^4)$ [in general $a' \sim (v/c)^2$], two additional states are possible, besides those already indicated. For $-f < a' < 0 < g'$ (state III_1), the vector γ lies in the (010) [or (100) plane at a small angle θ to the crystal axis, with $\cos\theta = \sqrt{-a'/f}$; then there is a spontaneous magnetic moment $m_0 = |e|\sqrt{1 + a'/f}/B$, directed along [100] (or [010]). In state III_2 $(-f < a' < g' < 0)$, γ and m lie in one of the (110) planes; m in this case has a component along the crystal axis. We shall not give the unwieldy formulas for θ and M_0 that apply to this case.

We now ask: between which of the states of the crystal mentioned above are phase transitions of the second kind possible? According to Landau's theory, a phase transition between two states can occur only if the symmetry group of the one state is a symmetry subgroup of the other. We write the symmetry elements that describe the magnetic symmetry classes of the various possible states of our crystal: *

I. $C_2,\ 2C_4R,\ 2U_2,\ 2U'_2R,\ I,\ \sigma_h,\ 2S_4R,\ 2\sigma_v,\ 2\sigma'_vR$
II_1. $C_2R,\ U_2R,\ U_2,\ I,\ \sigma_hR,\ \sigma_vR,\ \sigma_v$
II_2. $C_2R,\ U'_2R,\ U'_2,\ I,\ \sigma_hR,\ \sigma'_vR,\ \sigma'_v$
III_1. $U_2,\ I,\ \sigma_v$
III_2. $U'_2R,\ I,\ \sigma'_vR.$

It is easily seen that only the following transitions of the second kind are possible:

$$I \longleftrightarrow III_1,\ III_2;\quad II_1 \longleftrightarrow III_1;\ II_2 \longleftrightarrow III_2.$$

Furthermore, it can be shown that in the expansions of the thermodynamic potential that correspond to these transitions, third-order terms are absent, so that according to Landau's theory the indicated transitions are actually allowed.

States III_1 and III_2 are realized, as has already been pointed out, in a very narrow temperature range where the coefficient a' is approaching zero; that is, in the ranges where transitions between states II_1 and I, or between states II_2 and I, occur. Let us consider, for example, how a continuous transition between states II_1 and I occurs. In state II_1 the spins of the two ions in the unit cell are arranged as is shown in Fig. 2a. At some temperature T'_c a transition occurs between states II_1 and III_1. With further change of temperature, the spins leave the (001) plane and begin to approach the crystal axis. The spontaneous magnetic moment thereupon decreases in magnitude, retaining the direction [100]. Finally, at some other temperature T''_c a continuous transition occurs from III_1 to I. At this point the spins of the ions lie along the direction of the crystal axis, and the spontaneous magnetic moment is equal to zero. Thus the transition between states II_1 and I occurs by two successive transitions of the second kind.

The measurements of Erickson, who found in the case of NiF_2 and angle of 10° between the spins and the axis, show that at T = 25°K the NiF_2 crystal is in one of the transitional states (III_1 or III_2). However, as was mentioned above, Erickson's experiment has too low precision. Measurements of Stout and Matarrese,[4] who observed a spontaneous magnetic moment in NiF_2, show that NiF_2 at this temperature cannot be in state I.

I express my thanks to E. M. Lifshitz for discussion of the work.

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*Here C_2 denotes a vertical twofold axis, C_4 a vertical fourfold axis, U_2 a horizontal twofold axis directed along [100] or [010], U'_2 a twofold axis directed along [100] or [1$\bar{1}$0], S_4 a fourfold reflection-rotation axis, σ_v and σ'_v vertical symmetry planes perpendicular respectively to axes U_2 and U'_2, σ_h a horizontal plane, I the operation of inversion, and R the operation of change of sign of the spins.

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ELECTROMAGNETIC RADIATION IN HIGH-ENERGY NUCLEAR INTERACTIONS

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Bremsstrahlung emitted by charged particles participating in high-energy nuclear interactions is computed on the basis of the hydrodynamical model of an elementary act. The radiated energy is found to be a constant fraction, whereas bremsstrahlung from point charges increases logarithmically with energy.

THE problem of the electromagnetic radiation which accompanies nuclear interactions has arisen frequently. Since the radiation from suddenly stopped charged particles increases logarithmically with the particle energy,[1] attempts were made to explain the soft component of cosmic radiation in this way at a time when the existence of π^0 mesons had not yet been established. The question has appeared again,* but now we have a better understanding of the structure of nuclear particles and of the character of high-energy nuclear interactions (there exists a consistent hydrodynamical theory of multiple particle production). The electromagnetic radiation from real particles participating in an elementary act can therefore be discussed without being confined to estimates[1] which, as we shall see, are correct only when the particles are assumed to be extremely small.

In the present article the electromagnetic radiation which appears in nuclear collisions due to the slowing down of extremely fast charged particles is investigated on the basis of Landau's[3] hydrodynamical model of nuclear interactions.

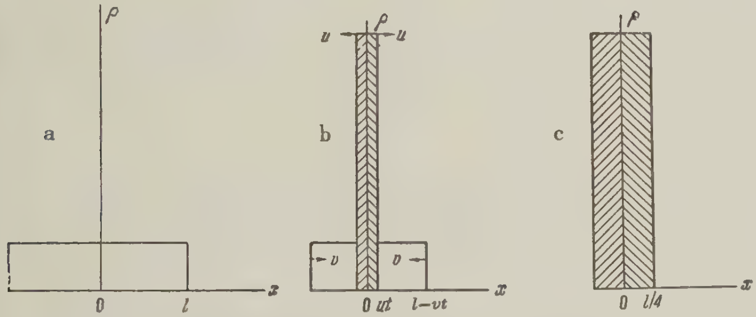

FIG. 1. "Stopping" stage: a — initial instant of collision; b — compression of the nuclei by the shock wave; c — instant when the shock wave has reached the edges of the nuclei.

For simplicity we shall consider the collision of two identical nuclei at high energies. In the center-of-mass system after contact the nuclear matter is symmetrically compressed by a shock wave and a hydrodynamical system is formed. As has been shown by Khalatnikov,[4] after the passage of the shock wave and the complete stopping of the nuclei symmetrical expansion begins which is first described by a traveling wave and then by a general one-dimensional solution. The formation of new particles takes place in the last, three-dimensional stage. The electromagnetic radiation which appears can obviously be estimated classically by assuming the charge to be distributed uniformly over the entire mass, with the charge density thus proportional to the mass density.

The energy radiated in an element of solid angle do and in the frequency interval $d\omega$ is

$$dW_{\mathbf{n}\omega} = c\,|\,[\mathbf{k}\times\dot{\mathbf{A}}_{\omega}]\,|^2 R_0^2\, do\, d\omega, \tag{1}$$

where $\mathbf{A}_\omega$ is obtained from the usual formula for retarded potentials:

$$\mathbf{A}_\omega = \frac{e^{ikR_0}}{cR_0}\int \mathbf{j}_\omega e^{-i\mathbf{kr}}\, dV,$$

and $\mathbf{j}_\omega$ is determined from the hydrodynamical solution.

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*The author is indebted to V. I. Zatsepin for calling her attention to Friedländer's article.[2]

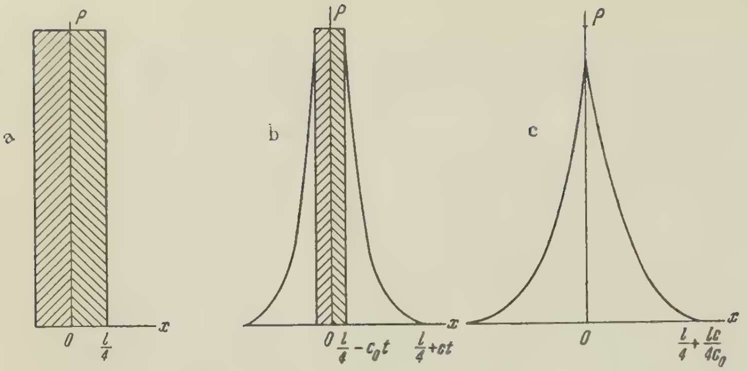

FIG. 2. Stage of "acquiring motion": a — the instant when the shock wave reaches the edges of the nuclei; b — passage of the rarefaction wave; c — the instant when the traveling waves meet.

Earlier calculations[1] show that energy is emitted principally at high frequencies. The characteristic frequency of the first stage (stopping) is determined by the transit time of the shock wave through the nucleus, which is $a\sqrt{1-\beta^2}/u$, where $u = c/3$ is the velocity of the shock wave and a is the diameter of the nucleus. In the expansion all of the matter is set into motion, mainly during the time required for the trailing edge of the traveling wave to reach the plane of symmetry from which expansion takes place. This time is $a\sqrt{1-\beta^2}/4c_0$, where $c_0 = c/\sqrt{3}$ is the velocity of the traveling wave. The time of the remaining stages is about $1/(1-\beta^2)$ greater and the accelerations are correspondingly smaller; it is therefore clear that radiation occurs principally during the first two stages.

We shall assume that the initial charge density ρ is Ze/V_0, where $V_0 = (\pi a^2/4)/a\sqrt{1-\beta^2}$ is the initial volume of the colliding nuclei. The current j differs from zero and is equal to ρv for all $|x|$ in the range $(\ell/4, \infty)$ when $-|x|/v < t < -|x|/v + \ell/v$, and for $\ell/4 > |x| > 0$ when $-|x|/v < t < |x|/u$ ($\ell/4 = a\sqrt{1-\beta^2}/4$ is the longitudinal dimension of the nuclei after compression by the shock wave). Behind the shock front the fluid is at rest and the current vanishes (Fig. 1).

In the traveling (rarefaction) wave

$$j_r = \frac{\rho c^2}{\sqrt{2}c_0}\left(\frac{c-c_0}{c+c_0}\right)^{c/2c_0}\frac{[c_0(t-t_0)+(x-x_0)]\,[c(t-t_0)-(x-x_0)]^{(c-c_0)/2c_0}}{[c(t-t_0)+(x-x_0)]^{(c+c)_0/2c_0}}$$

where $t_0 = \ell/4u$ and $x_0 = \ell/4$ are the initial coordinates of the front of the rarefaction wave. This current differs from zero for

$$l/4 < |x| < 0 \text{ and } |x|/c_0 + l/4u < |t| < l/4c_0 + l/4u$$

and for

$$l/4 < |x| < l/4 + lc/4c_0 \text{ and } |x|/c + l/4u < |t| < l/4c_0 + l/4u$$

(Fig. 2).

From this definition of the current we obtain j_ω and then A_ω:

$$A_\omega = \frac{\rho\beta ac^2}{R_0\omega\sin\vartheta}J_1\left(\frac{a\omega}{2c}\sin\vartheta\right)\left\{\frac{(1+3\beta)}{\omega^2(3+\cos\vartheta)(1-\beta\cos\vartheta)}\left[2\sin^2\frac{a\omega(3+\cos\vartheta)}{8c} - i\sin\frac{a\omega(3+\cos\vartheta)}{4c}\right] + \frac{ce^{i\omega(t_0-x_0/c)}}{2\sqrt{2}\,c_0}\left(\frac{c-c_0}{c+c_0}\right)^{c/2c_0}\right.$$
$$\left.\times\int\limits_{-c_0/c}^{1}\varphi(\xi)\left[\frac{\exp\{i\omega(1-\xi\cos\vartheta)\,l/4c_0\}-1}{\omega^2(1-\xi\cos\vartheta)^2} - i\,\frac{\exp\{i\omega(1-\xi\cos\vartheta)\,l/4c_0\}}{\omega(1-\xi\cos\vartheta)}\right]d\xi\right\} + A_\omega^-, \quad (2)$$

$$\varphi(\xi) = \frac{(1-\xi)^{(c-c_0)/2c_0}(c_0/c+\xi)}{(1+\xi)^{(c+c_0)/2c_0}}.$$

In Eq. (2) the expression within the curly brackets corresponds to radiation to the right; the first term within the brackets pertains to radiation from the left-hand shock wave (the first stage; for an undeformable "point" charge we obtain an analogous term); the second term gives the radiation from the right-hand traveling wave. A symmetrical expression for radiation to the left is obtained by changing the sign of $\cos\vartheta$. Equation (2) shows that the radiation is concentrated within small angles [because the denominators contain the factors $(1-\beta\cos\vartheta)$ and $(1-\xi\cos\vartheta)$]. Since we will ultimately be interested in radiation in the laboratory system and the center of mass is regarded as moving from left to right in the laboratory system, we shall neglect radiation to the left and A_ω^- will hereinafter be dropped.

When the traveling wave is taken into account the expression for the energy contains terms of the order of $(1-\beta^2)$. This results from the fact that whereas in the first stage the entire charge is slowed down from its relativistic velocity, in the second stage it acquires motion gradually, and only a small portion (close to the front of the traveling wave) acquires a velocity $\sim c$ during this period. Thus the entire radiation is determined by the stopping stage and we have

$$W \approx \frac{4^5 (Ze)^2 c^3}{2\pi a^4 (1-\beta^2)} \int_0^\infty \int_0^\pi f(\vartheta) \frac{J_1^2 (a\omega \sin\vartheta/2c) \sin^2 (l\omega (3+\cos\vartheta)/8c) \sin\vartheta d\vartheta d\omega}{\omega^4 (1-\beta^2 \cos^2\vartheta)^2}, \quad f(\vartheta) = (1+\cos\vartheta)^2/(1+3\cos\vartheta)^2. \tag{3}$$

Making the substitutions $\sin\vartheta/\sqrt{1-\beta^2} = y$ and $\omega\ell/2c = z$ together with the assumption $1/\sqrt{1-\beta^2} = \infty$, we find that up to terms containing $(1-\beta^2)$ the radiated fraction of the energy in the laboratory system is

$$\Delta \approx (13\alpha Z^2/\pi A^{4/3})\, \mu/M \tag{4}$$

where the factor μ arises from $a \approx (\hbar/\mu c)A^{1/3}$ and M from $\sqrt{1-\beta^2} = MAc^2/E$. For proton collisions $Z = A = 1$ and $\Delta \approx 0.4\%$.

This value of the radiated fraction of the energy is much smaller than was estimated for point particles.[1] The difference is explained by the fact that Eq. (4) does not contain the large logarithmic factor $-\ln\sqrt{1-\beta^2}$. This factor disappears because the integrand in (3) contains the function $J_1^2(\omega a \sin\vartheta/2c)$ which oscillates rapidly at high frequencies and which arose from the extent of the nuclei in the transverse direction. In the center-of-mass system the particles have the shape of discs because of the Lorentz contraction. At high frequencies the radiation from different elements of a slowed-down disc will interfere and give a sharp maximum only at angles $< \ell/a \approx \sqrt{1-\beta^2}$ around the direction of motion. However, it is easily seen that in the earlier formulas[1] the principal term $-\ln\sqrt{1-\beta^2}$ was derived from the angular range $1 \gg \vartheta \gg \sqrt{1-\beta^2}$.

The former result is obtained from (3) if we make a approach zero. In order to obtain the final result the integral over ω must, as usual, be cut off at a certain $\omega_{\max}$:

$$W_{l\to 0} \sim \int_0^{\omega_{\max}} d\omega \int_0^\pi \frac{\sin^3\vartheta d\vartheta}{(1-\beta\cos\vartheta)^2} \sim \alpha\omega_{\max}\left(\ln\frac{1}{\sqrt{1-\beta^2}} + \text{const}\right). \tag{5}$$

In our expression (3) for the energy the integral converges because of the interference term $J_1^2(\omega a \times \sin\vartheta/2c)$ at frequencies ω_c for which $\omega_c a\sqrt{1-\beta^2}/2c \sim 1$. These frequencies are present in (3) because for $a = \hbar/\mu c$ we have

$$\omega_c \simeq \mu c^2/\sqrt{1-\beta^2} < E,$$

where E is the energy of the colliding particles in the center-of-mass system.

For particles with dimensions much smaller than their Compton wavelength it would be necessary to introduce the cutoff at $\omega_c \sim E$. Then (3) for the given ω_c would contain a logarithmic region of integration for the angles $\sqrt{1-\beta^2} \ll \vartheta \ll (\lambda_0/a)\sqrt{1-\beta^2}$ (λ_0 is the Compton wavelength), which would lead to $\ln(\lambda_0/a)$, thus somewhat increasing the result.

The hydrodynamical procedure developed here is inapplicable for electrons, and evidently also for muons, because of their weak interactions. The radiation must then be determined from the usual formulas,[1] which contain a logarithmic factor. The applicability of these formulas is determined by the possibility of using perturbation theory.

The author is grateful to E. L. Feinberg for suggesting this research and to I. L. Rozental' for valuable discussions.

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THEORY OF CYCLOTRON RESONANCE IN METALS

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The surface impedance of a metal is calculated, with arbitrary dispersion law and collision probability, for the case of radio-frequency and constant magnetic fields both parallel to the metal surface. The field-dependence and temperature-dependence of the surface impedance are studied over the whole range of possible field strengths. It is explained how one can determine the form of the Fermi surface, the velocity of electrons in the surface, and the probability of electron transitions caused by collisions. The predictions of the theory are compared with experiments on cyclotron resonance in metals.

1. INTRODUCTION

THE authors[1,2] predicted and investigated theoretically the phenomenon of cyclotron resonance in metals. This resonance differs basically from the well-known diamagnetic resonance[3] in semiconductors in two respects: (1) Cyclotron resonance occurs not only when the imposed frequency ω is equal to the cyclotron frequency Ω, but also at the multiple frequencies $\omega \approx \Omega, 2\Omega, \ldots,$ whether the metal is isotropic or not. (2) Cyclotron resonance in metals is possible only when the constant magnetic field **H** is strictly parallel to the metal surface. A small angle between the magnetic field and the surface makes the resonance disappear.

Cyclotron resonance has recently been observed in tin by Fawcett,[4] Bezuglyi and Galkin,[5] and Kip, and in bismuth by Dexter and Lax[6] and Aubrey and Chambers.[7] At the same time the variation of the high-frequency surface resistance of tin, copper, and bismuth in strong magnetic fields, predicted by us earlier,[8] has been confirmed.

We are concerned now with four problems.

(1) To compare the experimental results with theory.

(2) To extend the theory to arbitrary field-strengths.

(3) To find the temperature-dependence of the impedance.

(4) To explain how the field-dependence of the surface impedance gives information about the Fermi surface* $\epsilon(\mathbf{p}) = \zeta$.

In Section 4 we show that in principle the experiments enable us to determine completely the shape of the Fermi surface and the velocity of the electrons in it.

It is particularly interesting to find that the geometry of the Fermi surface (especially the main components of it) has a qualitative effect on the behavior of cyclotron resonance. This allows us to make some deductions about the Fermi surface from a rapid look at the experimental results, without any complicated analysis of the resonance curves.

2. FIELD AND TEMPERATURE DEPENDENCE OF THE SURFACE IMPEDANCE

The main objective of our work is to determine and study the complete surface impedance tensor of a metal in radio-frequency and constant magnetic fields both parallel to the surface. Anomalous skin-effect conditions are assumed ($\delta \ll r, \ell, v/\omega$). The surface impedance tensor $Z_{\mu\nu}$ expresses the relation between the electric field $\mathbf{E}(0)$ at the metal surface ($z = 0$) and the total current **I** following in the metal

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*Here ϵ is the energy and **p** the wave-vector of an electron, ζ is the chemical potential of the electron gas in the metal, and $\epsilon(\mathbf{p})$ is an even periodic function of **p** with the reciprocal lattice for period.

$$E_\mu(0) = \sum_{\nu=1}^{2} Z_{\mu\nu} I_\nu = \frac{ic^2}{4\pi\omega} \sum_{\nu=1}^{2} Z_{\mu\nu} E'_\nu(0), \quad Z_{\mu\nu} = R_{\mu\nu} + iX_{\mu\nu} \quad (\mu, \nu = x, y). \tag{2.1}$$

The problem involves four quantities with the dimensions of a length, the Larmor radius r, the effective electron mean free path $\ell_{\text{eff}} = vt_0/(1 + \omega t_0)$, the actual mean free path $\ell = vt_0$ (v is the electron velocity and t_0 the mean collision time), and the skin-depth δ. Convenient asymptotic expressions for the surface impedance are obtainable only in the limits of very weak fields ($r \gg \ell^2/\delta$) and of fairly strong fields ($r \leq \ell$). We shall exhibit the final formulae for both cases, taking the results from Eq. (3.1) and (5.8) — (5.10) of Ref. 2.

1. Weak Fields

For simplicity we assume a quadratic dispersion law $\epsilon = p^2/2m$, where m is an effective mass. Then

$$Z(H) = Z^{(0)}\left(1 + \frac{3}{128}\frac{\sigma^* l^* Z^{(0)}}{\gamma^2}\right), \quad Z^{(0)} = \left(\frac{\sqrt{3}\pi\omega^2 l}{c^4\sigma}\right)^{1/3}(1 + i\sqrt{3}), \tag{2.2}$$

where $Z^{(0)}$ is the surface impedance in zero field, $\sigma^* = \sigma/(1 + i\omega t_0)$, σ is the static conductivity of the metal,

$$l^* = l/(1 + i\omega t_0), \quad \gamma = r/l^* = i\omega/\Omega + 1/\Omega t_0 = H_0/H, \quad \Omega = eH/mc.$$

The small parameter, in which Eq. (2.2) is an expansion, is $[\sigma^* \ell^* Z^{(0)}/\gamma^2]$. The effect of the magnetic field becomes noticeable as soon as $H \geq H_1$, with $H, H_1 \ll |H_0|$. This is natural, since the value

$$H_1 = |H_0/[\sigma^* l^* Z^{(0)}]^{1/2}| \ll |H_0|,$$

defines the field strength at which the arc length $s = \sqrt{r\delta_{\text{eff}}}$ of an electron path in the skin becomes comparable with the effective mean free path ℓ_{eff}. Apart from a numerical factor, Eq. (2.2) remains valid with any dispersion law and any direction of the magnetic field relative to the metal surface.

2. Strong Fields (either resonant or non-resonant)

$$Z_{\mu\nu} = 2(\sqrt{3}\pi\omega^2/c^4)^{1/3} e^{i\pi/3} (A^{-1/3})_{\mu\nu}, \quad Z_{xx,\,yy} = 2(\sqrt{3}\pi\omega^2/c^4)^{1/3} e^{i\pi/3} [(\xi_1 + \xi_2)\xi_1\xi_2 + A_{xx,\,yy}]/\xi_1\xi_2(\xi_1^2 + \xi_1\xi_2 + \xi_2^2),$$
$$Z_{xy} = Z_{yx} = 2(\sqrt{3}\pi\omega^2/c^4)^{1/3} e^{i\pi/3} A_{xy}/(\xi_1^2 + \xi_1\xi_2 + \xi_2^2), \tag{2.3}$$

with

$$\xi_{1,2} = \left\{\frac{1}{2}(A_{xx} + A_{yy}) \pm \left[\left(\frac{A_{xx} - A_{yy}}{2}\right)^2 + A_{xy}^2\right]^{1/2}\right\}^{1/3}, \quad A_{\mu\nu} = \frac{8e^2}{3h^3}\int_0^{2\pi} \frac{n_\mu(\varphi)\, n_\nu(\varphi)}{K(\varphi)} \frac{d\varphi}{1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau_0)}, \tag{2.4}$$

The cube roots are to be taken with arguments in the interval $(-\pi/6, \pi/6)$; this is always possible since the radicands have positive real parts. The real part of the impedance is then necessarily positive. The notations are: e the electron charge, h Planck's constant, $\mathbf{v} = v\mathbf{n}$ the electron velocity, K the absolute value of the Gaussian curvature of the Fermi surface, $\Omega = eH/mc$ the cyclotron frequency, $m = (1/2\pi)\partial S/\partial\epsilon$ the effective mass of an electron, $S(\epsilon, p_x)$ the area of intersection of the surface $\epsilon(\mathbf{p}) = \epsilon$ by the plane† $p_x = \text{const}$. The integration in Eq. (2.4) is along the curve $v_z = 0$ on the surface $\epsilon(\mathbf{p}) = \zeta$, φ is the angle between the velocity (i.e., the normal to the Fermi surface) and the direction of the constant magnetic field H, and $1/\tau_0 = \overline{(1/t_0)}$ is the average along the path of integration of the rate of collision of electrons with impurities, phonons, electrons, lattice defects, surface irregularities, etc.

In general, the complex tensors $A_{\mu\nu}$ and $Z_{\mu\nu}$ cannot be reduced to diagonal form by a rotation of axes. But if such a reduction is possible (for example, if $R_{\mu\nu}$ and $X_{\mu\nu}$ are proportional) then the principal values of Z_α are given by

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†The x axis is taken parallel to H, so that p_x is the component of the wave vector parallel to the constant magnetic field.

$$Z_\alpha = 2(\sqrt{3}\pi\omega^2/c^4)^{1/3}e^{i\pi/3}A_\alpha^{-1/3} \tag{2.5}$$

where A_α are the principal values of $A_{\mu\nu}$.

(a) Resonant Region. The dependence of the surface impedance on magnetic field and frequency in the resonant region has been already discussed by us in detail.[1,2] We have shown that the form of the dispersion law has an important effect on the shape of the resonance curve. Resonance occurs at values of the effective mass $(1/2\pi)(\partial S/\partial\epsilon)_{\text{ext}}$ which are extremal with respect to φ. An extremum of the effective mass will in any case occur at the central section $p_x = 0$, $\epsilon = \zeta$, and also at elliptical points of support of the Fermi surface* (i.e., points at which the electron velocity is parallel to the constant magnetic field; such points have $m = 1/v\sqrt{K}$).

It can be shown that a resonant behavior (i.e., a minimum of R and X), will be seen only in the $Z_{x'x'}$ component. Here x' means the direction of H when the resonance corresponds to the effective mass at an elliptical point of support, and x' is the direction of the velocity at the point $v_z = 0$ on a central section of the Fermi surface when the resonance corresponds to the effective mass on a central section.

The absorbed power P is given by

$$P = \frac{c^2}{8\pi}\{R_{xx}(E_x^{(i)})^2 + R_{yy}(E_y^{(i)})^2\} \tag{2.6}$$

($E^{(i)}$ is the field amplitude of the incident wave) and has a resonant behavior only when the incident wave is plane polarized in the plane of x'. More exactly, the angle between $E^{(i)}$ and x' should not exceed $\chi \sim [R^{\text{res}}/R^{(0)}]^{1/2}$. In the case of a maximum effective mass $\chi \lesssim (q^2/\omega\tau_0)^{1/12}$, and in case of a minimum $\chi \lesssim (q^2/\omega\tau_0)^{2/9}$. Here $q = \omega/\Omega_{\text{res}}$ is an integer.

If a resonance corresponds to an extremal effective mass which is neither equal to the mass at a central section nor to the mass at an elliptical point of support, then all components of the tensor $Z_{\mu\nu}$ have resonant behavior. Resonant absorption of power will then occur for any polarization of the electric field in the incident wave. The formulae for all components of the impedance which have resonant behavior have been exhibited previously[1,2] for frequencies close to resonance.

(b) Nonresonant Region. When the magnetic field is so strong that

$$\Omega \gg 2\pi/\tau_0,\ 2\pi\omega,\ \pi\omega\cdot\omega\tau_0 \tag{2.7}$$

the tensor $A_{\mu\nu}$ can be reduced to the principal axes (x'', y''), and its principal values are given by

$$A_\alpha = \frac{8e^2}{3h^3}(i\omega + 1/\tau_0)^{-1}\,\overline{(n_\alpha^2\Omega/K)} \tag{2.8}$$

where the bar indicates an average with respect to the angle φ. Hence Eq. (2.5) gives

$$Z_\alpha(H) = 2\left(\frac{\sqrt{3}\pi\omega^2}{c^4}\right)^{1/3} e^{i\pi/3}\left[\frac{8e^2}{3h^3}\overline{\left(\frac{n_\alpha^2\Omega}{K}\right)}\frac{\tau_0}{1 + i\omega\tau_0}\right]^{-1/3} \sim H^{-1/3}\omega^{2/3}(1 + \omega^2\tau_0^2)^{1/6}\exp\left[\frac{i}{3}(\pi + \tan^{-1}\omega\tau_0)\right] \tag{2.9}$$

One of us[8] obtained this formula earlier in the case of a quadratic dispersion law. In particular, when the frequency is low $(\omega\tau_0 \ll 1)$

$$Z_\alpha = \left(\frac{\sqrt{3}\pi\omega^2}{c^4}\right)^{1/3}\left[\frac{8e^2\tau_0}{3h^3}\overline{\left(\frac{n_\alpha^2\Omega}{K}\right)}\right]^{-1/3}(1 + i\sqrt{3}) \sim \left(\frac{\omega^2}{H\tau_0}\right)^{1/3}(1 + i\sqrt{3}), \tag{2.10}$$

while at high frequencies $(\omega\tau_0 \gg 1)$

$$Z_\alpha = 2\left(\frac{\sqrt{3}\pi}{c^4}\right)^{1/3}\left[\frac{8e^2}{3h^3}\overline{\left(\frac{n_\alpha^2\Omega}{K}\right)}\right]^{-1/3}\left(\frac{1}{3\tau_0} + i\omega\right), \quad R_\alpha \sim H^{-1/3}/\tau_0,\ X_\alpha \sim \omega H^{-1/3}. \tag{2.11}$$

In the case of high frequencies $(\omega\tau_0 \gg 1)$, the formulae are still simple in the range

$$2\pi\omega \ll \Omega \ll \pi\omega\cdot\omega\tau_0. \tag{2.12}$$

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*Resonance occurs in this case only when the frequency is not too high. The skin depth has to be small compared with the Larmor radius, which itself becomes small near to an elliptical point of support.

where the magnetic field is of intermediate strength. The result is then

$$A_{\mu\nu} = -i\,\frac{4e^2}{3\pi h^3\omega}\int_0^{2\pi}\frac{n_\mu(\varphi)\,n_\nu(\varphi)}{K(\varphi)}\Omega(\varphi)\,d\varphi + \frac{4e^2}{3h^3}\int_0^{2\pi}\frac{n_\mu(\varphi)\,n_\nu(\varphi)}{K(\varphi)}\,d\varphi, \tag{2.13}$$

and the impedance is independent of τ_0,

$$R_{\mu\nu} \sim \omega^2 H^{-4/3}, \quad X_{\mu\nu} \sim \omega H^{-1/3}. \tag{2.14}$$

The temperature dependence of the surface impedance can be obtained from the results of Refs. 1 and 2 and the present paper by substituting

$$1/\tau_0 = 1/\tau_0^{\mathrm{res}} + (1/\tau_0^{\mathrm{p}})(T/\Theta)^3 + \frac{1}{\tau_0^e}(T/\Theta_e)^2, \tag{2.15}$$

This behavior of τ_0 holds for $\hbar\omega \ll kT$; τ_0^{res} is the relaxation time corresponding to the residual resistivity, τ_0^{p} and τ_0^{e} the relaxation times for scattering by phonons and electrons, Θ is the Debye temperature, and Θ_e is the characteristic temperature for electron-electron collisions. Equation (2.15) is valid because of the possibility of defining a mean time of free passage $t_0(p)$ in the theory of the anomalous skin effect.[1,2]

In particular, when phonon collisions are preponderant, we have the following results:

(1) Quadratic dispersion law,

$$\begin{gathered} R_{\mathrm{res}} \sim (T/\Theta)^2, \ |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^{3/2}, \\ X_{\mathrm{res}} \sim T/\Theta, \quad |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^3, \\ (X/R)_{\mathrm{res}} \sim (T/\Theta)^{-3/2}, \ |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^{3/2}; \end{gathered} \tag{2.16}$$

(2) Nonquadratic dispersion law, $\partial S/\partial\epsilon$ having a minimum,

$$\begin{gathered} R_{\mathrm{res}} \sim (T/\Theta)^{4/3}, \quad |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^2, \\ X_{\mathrm{res}} \sim (T/\Theta)^{1/3}, \ |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^3, \\ (X/R)_{\mathrm{res}} \sim (T/\Theta)^{-1}, \quad |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^2; \end{gathered} \tag{2.17}$$

(3) Nonquadratic dispersion law, $\partial S/\partial\epsilon$ having a maximum,

$$Z_{\mathrm{res}} \sim (T/\Theta)^{1/2}, \ |\omega - q\,\Omega_{\mathrm{res}}| \sim (T/\Theta)^3, \ \Omega_{\mathrm{res}} = eH_{\mathrm{res}}/cm_{\mathrm{ext}} \approx \omega/q \tag{2.18}$$

with q an integer.

(4) Strong field, $(\Omega \gg 2\pi\omega,\ \pi\omega^2\tau_0,\ 2\pi/\tau_0)$. For $\omega\tau_0 \ll 1$, Z_α is proportional to temperature. For $\omega\tau_0 \gg 1$, $R_\alpha \sim (T/\Theta)^3$ and X_α is independent of temperature.

(5) Intermediate field, $(2\pi\omega \ll \Omega \ll \pi\omega^2\tau_0)$. At high temperatures the impedance is independent of temperature.

Naturally, the above results are valid only so long as the skin effect is anomalous, i.e., for $\delta \ll r,\ |l^*|$. This means that the frequencies are restricted by

$$c^2/2\pi\sigma l^2 \ll \omega \ll (v/c)\sqrt{2\pi ne^2/m}$$

and the fields by

$$H \ll v[2\pi nm\omega\tau_0/(1+\omega\tau_0)]^{1/2}$$

where n is the electron density in the metal. At low temperatures, taking $\tau_0 \sim 10^{-11}$ sec, $n \sim 10^{22}\ \mathrm{cm}^{-3}$, $m \sim 10^{-27}$ g, these limits become $10^7\ \mathrm{sec}^{-1} \ll \omega \ll 10^{13}\ \mathrm{sec}^{-1}$, $H \ll 10^6[\omega\tau_0/(1+\omega\tau_0)]^{1/2}$ oersted.

3. COMPARISON BETWEEN THEORY AND EXPERIMENT

In the introduction we mentioned that the variation of surface resistance with magnetic field has been observed in three metals, tin,[4,5] copper,[4] and bismuth[6,7] (see Figs. 1 and 2). In all cases there is qualitative agreement between experiment and theory. (1) A resonance is observed, except in the case of Fawcett's experiment with copper, in which the quantity $\omega\tau_0$ is of the order of unity and is not large enough for a resonance. (2) The surface resistance decreases monotonically in strong magnetic fields; the increase of surface resistance in bismuth for H > 2000 oersted is probably due to the onset of nor-

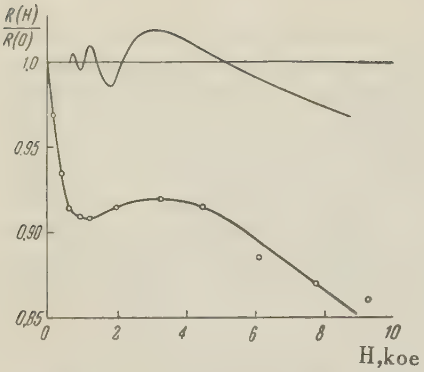

FIG. 1. Cyclotron resonance in tin according to Fawcett. Figure taken from Ref. 12. Experimental points for $\omega/2\pi = 24 \times 10^9$, $T = 4.2°$ K, $\omega\tau_0 \sim 27$ are marked ○. The upper curve is theoretical.

mal skin-effect conditions when $r < \delta$. (3) R(H) is slightly decreasing in weak fields. (4) The resonant frequency is displaced as the temperature decreases,[5] because of the increase in ωt_0. (5) The surface resistance is anisotropic, even in the case of a single crystal of copper which possesses cubic symmetry, the axis of the asymmetry being provided by the direction of the constant magnetic field.

Nevertheless, the experimentally observed resonance is less deep, and the decrease of surface resistance in strong fields is smoother, than the theory predicts.

In the case of bismuth, the discrepancy between theory and experiment is apparently connected with the anomalously small density and effective mass of the electrons, which cause the basic assumption $\delta \ll r$ to be violated.

In the case of tin and copper, we believe that the cause of the discrepancy is as follows. The theoretical formulae apply only when two conditions are satisfied.

(1) For any value of the magnetic field, the angle between the field and the metal surface should satisfy

$$\Phi \ll (r/l)(\delta/r)^{2/3} = (\delta^2 r/l^3)^{1/3} \sim H^{-1/3}. \quad (3.1)$$

The larger the field, the more strictly must the directions be parallel. When Eq. (3.1) is not satisfied, the effective distance which an electron can travel while remaining in an appreciable electric field increases more slowly with magnetic field than it does when $\Phi = 0$, (see Fig. 3). The variation of the impedance with magnetic field is therefore smoother when Eq. (3.1) does not hold, and for sufficiently high fields the impedance will saturate. Fawcett[4] observed just such a smoothed-out variation of R(H) for copper and tin, using an angle Φ of the order of one degree, whereas in a field $H \sim 10^4$ oersted with

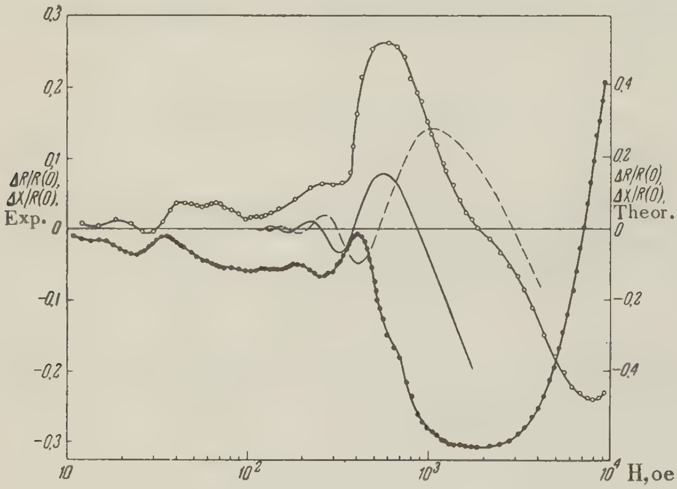

FIG. 2

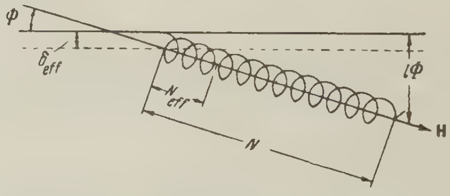

FIG. 3

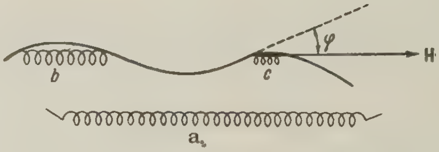

FIG. 4

FIG. 2. Cyclotron resonance in bismuth according to Aubrey and Chambers.[7] Full circles show $\Delta R/R(0)$ against H, open circles show $\Delta X/R(0)$ against H, for $\omega = 2\pi \times 9 \times 10^9$ cycles, $T = 4°$ K. The full curve gives theoretical values of $\Delta R/R(0)$, the dotted curve theoretical values of $\Delta X/R(0)$. $\Delta R = R(H) - R(0)$, $\Delta X = X(H) - X(0)$.

FIG. 3. Path of an electron in a magnetic field making an angle Φ with the metal surface. N_{eff} is the effective number of revolutions of an electron within the skin depth; N is the total number of revolutions between collisions.

FIG. 4. The "surface" mean free path of an electron. φ is the angular inclination of a surface irregularity. a shows an electron path in the interior of the metal, b and c are paths of electrons near the metal surface.

$\omega t_0 \sim 27$ the required upper limit for Φ is of the order of a few minutes. The inclination of the magnetic field will also produce similar effects in the resonance region.

(2) The second possible cause of departure of experimental results from theory is the imperfection of the metal surface. The collision rate $(1/\tau_0)$ which determines the depth of the resonance is an averaged rate of collisions for electrons close to the metal surface, since only these electrons contribute significantly to the current. The "surface" mean free time t_0^s may be quite different from the mean free time t_0^m in the interior of the metal which is calculated from the static conductivity (see Fig. 4). For t_0^s and t_0^m to be comparable, the angle of inclination φ of surface irregularities which are not small compared with the skin depth must also satisfy Eq. (3.1). Figure 4 shows how, if φ does not satisfy Eq. (3.1), δ_{eff}, and hence also t_0, will depend on H. In this case the dependence of the impedance on magnetic field is caused only by the fraction of the metal surface which is effectively parallel to the magnetic field, and this fraction in turn depends on H. Thus the values of ωt_0^m can be much larger than the values of ωt_0^s which determine the depth of the resonance. This is probably the explanation of the large width of the resonance in Fawcett's experiment,[4] in which $\omega t_0 \sim 27 \gg 1$.

Finally, it is possible that extremum values of m will occur only at points of support and at central sections of the Fermi surface. In this case one will observe sharp resonances only by using incident radiation with a definite polarization (see the preceding section).

4. THE POSSIBILITY OF DETERMINING THE FERMI SURFACE

The determination of the Fermi surface $\epsilon(\mathbf{p}) = \zeta$ and of the electron velocities in it is the basic problem of the electronic theory of metals. The problem is extremely difficult, partly because hitherto we have known nothing about the general shape of the surface except that it has central symmetry and the periodicity of the reciprocal lattice. The surface might be multiply connected, intersecting itself, or open. This makes difficult and partly ambiguous the theoretical interpretation even of those experiments[9] which allow in principle a determination of the form of the surface and of the velocities in it. It is therefore desirable to begin by exploring the topology of the surface, to find for example the number of nonsingular components into which it can be separated.

The results of Sec. 2 and of Refs. 1 and 2 show that the study of the anisotropy of cyclotron resonance leads to a number of conclusions about the geometry of the Fermi surface.

(1) If for every direction of the constant magnetic field and every polarization of the incident radiation we observe a resonance with the relative depth $(R_{\text{res}}/R^{(0)}) \sim (\omega\tau_0/2\pi q)^{-2/3}$, this resonance is evidently produced by ellipsoids. The number of ellipsoids is equal to the number of fundamental frequencies $(\omega \approx \Omega)$ corresponding to resonances of this type.

(2) If, for some directions of the constant magnetic field and for some polarizations of the incident radiation, the relative depth of a resonance has the behavior $(R_{\text{res}}/R^{(0)}) \sim (\omega\tau_0/q^2)^{-4/9}$ or $(R_{\text{res}}/R^{(0)}) \sim (\omega\tau_0/q^2)^{-1/6}$, then one or more components of the surface $\epsilon(\mathbf{p}) = \zeta$ are not ellipsoids. The resonance is produced by an extremal value of the effective mass, which is a minimum in the first case and a maximum in the second.

(3) The number of fundamental resonant frequencies with the polarization parallel to the constant magnetic field is in general greater than the number appearing with any other adjacent direction of polarization. The difference between these two numbers of fundamental frequencies is equal to one half the number of elliptical points of support of the Fermi surface corresponding to the given direction H. As the direction of the constant magnetic field varies, the minimum number of elliptical points of support cannot be less than twice the number of closed convex components of the surface. In particular, if for any direction of H the number of elliptical points of support is zero, then the surface $\epsilon(\mathbf{p}) = \zeta$ has no closed convex component. It may happen that there are no points of support in this direction, so that the surfaces are all open, or it may be that there are points of support of hyperbolic of parabolic type. If one knows the number of elliptical points of support in every direction, one may be able to deduce the number of nonsingular components of the surface $\epsilon(\mathbf{p}) = \zeta$, and the number of open sections.

(4) If, for a certain direction of the constant magnetic field, the number of fundamental resonant frequencies is greater for one direction of plane polarization of the incident radiation (not parallel to the direction of H) than for other directions of polarization, then one component of the surface $\epsilon(\mathbf{p}) = \zeta$ is centrally symmetric. The number of centrally symmetric components is equal to the maximum number of such exceptional directions of polarization which appear as the direction of the field is varied.

If, for some direction of the constant magnetic field, the number of exceptional directions of polarization (not coinciding with the direction of H) which have the maximum number N of fundamental frequencies is less than the number M of centrally symmetric components of the surface, then the difference (N − M) gives the number of open central sections perpendicular to that direction of the constant field. By varying the direction of the field we may find all the open central sections.

(5) If, for all directions of H, resonance occurs only with particular polarizations of the incident radiation, the resonance arises only from central sections and elliptical points of support. The effective mass then never attains its extremum with respect to φ. The extremum with respect to p_x is attained only at central sections.

Such a study of the geometry of the Fermi surface will obviously require very laborious experiments. One needs to observe the anisotropy of the impedance with respect to both the direction of the constant field and the polarization of the incident radiation. However, these difficulties arise from the complicated geometry of the Fermi surface itself. We wish to emphasize that the study of cyclotron resonance allows us to elucidate the structure of the main components of the conduction band of the metal. The clues to the structure of the Fermi surface, which are obtained from observations of the anisotropy and relative depth of resonances, may greatly facilitate the solution of the next problem to which we now turn, the quantitative determination of the surface.*

A given resonant frequency corresponds in general to a value of the effective mass either on a single centrally symmetric surface or on a pair of surfaces which together have central symmetry. It is therefore sufficient to consider the determination of a single surface. If this surface turns out not to have central symmetry, a second surface can be obtained from it by reflection in the center. For simplicity we suppose that the surface is convex.

Measurement of the resonant frequencies and of the values of R and X at resonance determines the quantities

$$\partial S/\partial\varepsilon, \quad K^2|\partial^3 S/\partial\varepsilon\,\partial\varphi^2|, \quad \tau_0$$

at points where $\partial S/\partial\epsilon$ is an extremum with respect to variations of φ. The anisotropy of the effect (see above) distinguishes the resonances arising from central sections and from elliptical points of support. In these two cases, the direction of polarization of the incident radiation for which resonance occurs determines the direction of the velocity at the corresponding point on the surface (see Sec. 2).

At an elliptical point of support $1/\tau_0 = 1/t_0(p)$, so that the collision rate at each point of the Fermi surface is directly determined, and $(\partial S/\partial\epsilon)_{\text{ext}} = 2\pi/v\sqrt{K}$. When we know $\partial S/\partial\epsilon$ and $\partial^3 S/\partial\epsilon\,\partial\varphi^2$, we can in many cases determine the form of the surface and the electron velocity in it. As an example we will discuss a convex surface with central symmetry. In this case, by observing the exceptional directions of polarization, not coinciding with the direction of H, for which resonance corresponding to a central section occurs, we determine the normal direction at every point on the surface. This allows us to fix the surface up to a similarity transformation. Knowing the shape of the surface, we deduce from $(\partial S/\partial\epsilon)_{\text{ext}}$ the magnitude of the velocities on the surface.[10] The scale of the similarity transformation can be determined, for example, from the value of X at resonance. This quantity fixes $K^2\,|\partial^3 S/\partial\epsilon\,\partial\varphi^2|$, or in the case of an ellipsoid $\overline{(1/K)}$, once τ_0 is known. A check on the consistency of the construction of the surface is provided by our knowledge of the values of $v\sqrt{K}$ at every point on it.

Measurements of the surface impedance off resonance† can also be used for finding the form of the Fermi surface and the electron velocities in it. Such measurements are less convenient, since the impedance is then produced by the combined effect of all components of the surface $\epsilon(\mathbf{p}) = \zeta$. We therefore discuss only the case of a one-component surface.

As we showed earlier,[2] from a knowledge of the principal values Z_α of the surface impedance under anomalous skin-effect conditions in zero magnetic field[11] we can determine the Gaussian curvature K at every point of the surface. If the surface is convex, its shape is uniquely determined by its Gaussian curvature.‡

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*The results stated here were in part published earlier.[2]

†Chambers[12] has recently arrived at similar conclusions by an independent method.

‡Note added in proof, (November 20, 1957). This requires the performance of experiments similar to those of Pippard.[13] After this paper was submitted for publication, we learned that Pippard has determined the Fermi surface of copper by this method (A. B. Pippard, Report to the 4th All-Union Conference on Low-Temperature Physics, Moscow, July 1957).

In addition, measurement of the impedance in strong magnetic fields ($\Omega \gg 2\pi/\tau_0$, $2\pi\omega$) at low frequencies ($\omega\tau_0 \ll 1$) allows us to determine $\overline{(\Omega\tau_0/K)}$. If we know the form of the surface and the velocities in it we can deduce τ_0. The values of $\overline{(\Omega\tau_0/K)}$ are obtained from Eq. (2.10), without assuming τ_0 to be independent of φ, by taking the average of $n_\alpha^2\Omega\tau_0/K$. When τ_0 is known, a similar argument determines $1/t_0(p)$, so that the probability for electron transitions from one state to another is in principle determined. We must emphasize that the value of $t_0(p)$ obtained in this way will agree with the value appropriate to bulk metal only when the conditions discussed in Sec. 2 are fulfilled. Measurement of the imaginary part of the impedance at intermediate field strengths ($2\pi\omega \ll \Omega \ll \pi\omega^2\tau_0$) and high frequencies ($\omega\tau_0 \gg 1$) determines

$$X_\alpha = \left(\frac{\sqrt{3}\pi\omega^2}{c^4}\right)^{1/3}\left[\frac{8e^2}{3h^3\omega}\overline{\left(\frac{n_\alpha^2\Omega}{K}\right)}\right]^{-1/3}$$

This gives an independent determination of $\overline{\Omega/K}$ and consequently of Ω/K at every point.

Finally, one can introduce data on the de Haas–Van Alphen effect to help determine the Fermi surface. Measurement of the periods of the "high-frequency" oscillations of diamagnetic susceptibility in very strong magnetic fields ($H > 10^5$ oersted) determines the extremal areas S_{ext} of the surface $\epsilon(p) = \zeta$. Experiments on cyclotron resonance in metals give values of $(\partial S/\partial\epsilon)_{ext}$, which are hard to obtain from the de Haas–Van Alphen effect. When we know both these quantities, we have a complete determination of the Fermi surface and of the velocities in it.

Unfortunately there is still a complete lack of the experimental data on cyclotron resonance which would enable us to study the dispersion law of conduction electrons.

CONCLUSIONS

(1) Cyclotron resonance occurs in metals at high frequencies ω and at low temperatures T, when the magnetic field is precisely parallel to the metal surface [$\Phi \ll (r/\ell)(\delta/r)^{2/3}$]. The resonance occurs at all the harmonic frequencies $\omega = \Omega, 2\Omega, \ldots$.

(2) The shape and width of the resonance, and the anisotropy of the effect, are strongly dependent on the shape of the Fermi surface $\epsilon(p) = \zeta$. If the surface is ellipsoidal, the resonance must occur for all directions of the magnetic field and for all polarizations of the radiofrequency field. The relative depth of the resonance is then given by

$$R_{res}/R^{(0)} \sim (\omega\tau_0)^{-2/3}, \quad X_{res}/X^{(0)} \sim (\omega\tau_0)^{-1/3}.$$

(3) If the surface is not ellipsoidal, the resonant frequency corresponds to a value of the effective mass $(1/2\pi)(\partial S/\partial\epsilon)_{ext}$ which either is extremal with respect to variations of p_x or is associated with an elliptical point of support of the surface.

(4) For a given direction of the magnetic field H, a resonance corresponding to an elliptical point of support occurs when the radio-frequency field is polarized parallel to H.

(5) A resonance corresponding to a central section occurs only when the polarization is parallel to the electron velocity at the point $p_x = v_z = 0$ on the surface. Here the normal to the metal surface is taken as the z-axis.

(6) A resonance corresponding to an extremum of the effective mass, on a section $p_x = \text{const.}$ of the surface $\epsilon(p) = \zeta$ which is neither a central section nor a degenerate section at a point of support, occurs for all direction of polarization of the radio-frequency field.

(7) The relative depth of the resonance is different for maximum and minimum values of the effective mass. For a maximum it is given by

$$R_{res}/R^{(0)} \sim (\omega\tau_0)^{-1/6}, \quad X_{res}/X^{(0)} \sim (\omega\tau_0)^{-1/6}.$$

and for a minimum by

$$R_{res}/R^{(0)} \sim (\omega\tau_0)^{-4/9}, \quad X_{res}/X^{(0)} \sim (\omega\tau_0)^{-1/6}$$

(8) Away from resonance, the form of the dispersion law does not affect the main theoretical conclusions. The dependence of the surface impedance on H, in not too weak magnetic fields [$H \gtrsim (mc/e) \times (1/\tau_0 + \omega)$] strictly parallel to the metal surface, is described approximately by the same formula which holds for a quadratic dispersion law:

$$Z = Z^{(0)}[1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau_0)]^{1/3}.$$

In particular, Eq. (2.9) holds for any dispersion law in very strong magnetic fields ($\Omega \gg 2\pi/\tau_0$, $2\pi\omega$, $\pi\omega^2\tau_0$). At high frequencies ($\omega\tau_0 \gg 1$) the dependence of the impedance on frequency and magnetic field is given by

$$R \sim \omega^2 H^{-4/3}, \quad X \sim \omega H^{-1/3}.$$

also in the intermediate range of field-strengths ($2\pi\omega \ll H \ll \pi\omega^2\tau_0$).

(9) In weak magnetic fields ($\Omega \ll \omega + 1/\tau_0$), the impedance is constant to within a few per cent. In particular, if the field is very weak ($H \ll H_1$), the impedance varies according to Eq. (2.2).

(10) The temperature dependence of the impedance is determined by the quantity $1/\tau_0$ given by Eq. (2.15). Here τ_0^{res}, τ_0^{p}, and τ_0^{e} are independent of temperature.

(11) When plane-polarized electromagnetic waves are reflected from a metal surface in a magnetic field, there is always a considerable rotation of the plane of polarization, and this rotation may show a resonant behavior.

(12) Experimental study of cyclotron resonance can give abundant information about the form of the Fermi surface and about the mean collision time $t_0(\mathbf{p})$ of the electrons. It will make possible: (a) a determination of the number of ellipsoidal components of the surface $\epsilon(\mathbf{p}) = \zeta$, (b) a test for the existence of non-ellipsoidal components of the surface, (c) establishment of a lower bound to the number of closed convex components, (d) determination of the number and orientation of open central sections, (e) determination of the number of centrally-symmetric components, (f) in many cases, a complete specification of the Fermi surface and of the electron velocities and mean collision times at every point of it (see Sec. 4).

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THEORY OF CYCLOTRON RESONANCE

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The possibility of cyclotron resonance is studied for a metal in a magnetic field which is inclined to the surface. The surface impedance for a metal in a magnetic field parallel to the surface is calculated for an arbitrary law of reflection of electrons from the surface.

1. INTRODUCTION

AZBEL' and the present author[1-3] worked out the theory of cyclotron resonance for the case of a metal in a magnetic field parallel to the surface, assuming the reflection of electrons from the surface to be purely diffuse. When the magnetic field is parallel and the dispersion law is quadratic, all the electrons in the top layer of the Fermi distribution contribute to the resonance, since they all have the same orbital revolution frequency Ω and they all pass many times within the skin depth. When the dispersion law has a more general character, only electrons with an extremal value of effective mass contribute to the resonance, and the relative depth of the resonance is then smaller than for a quadratic dispersion law. In an inclined magnetic field the great majority of electrons enter the skin depth only once, after which they disappear into the interior of the metal, and they therefore do not contribute to the resonance. However, Chambers[4] pointed out that, in a magnetic field inclined at a considerable angle Φ to the surface, a resonance can still be produced by the minority of electrons which have $\bar{v}_H \approx 0$ and therefore return to the skin layer many times without drifting along the direction of the field. The bar over $\bar{v}_H$ denotes an average along the path of an electron with $\epsilon(\mathbf{p}) = \zeta$, $p_H = \mathrm{const}$. Here ϵ is the energy, $\mathbf{p}$ the wave vector, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ the velocity, ζ the chemical potential of an electron. p_H and v_H are the projections of $\mathbf{p}$ and $\mathbf{v}$ along the direction of the magnetic field.

We showed earlier[2,3] that to the lowest order in $(\delta_{\mathrm{eff}}/r)$ the surface impedance Z of the metal is in general independent of the magnetic field. Field-dependent and resonant behavior occurs in general only in the next order in $(\delta_{\mathrm{eff}}/r)$. Here δ_{eff} is the effective skin depth and r the orbital radius of an electron in the magnetic field. It is important to calculate the magnitude of resonant effects in an inclined magnetic field and to see whether such effects might be observed experimentally.

It is also interesting to see how the law of reflection of electrons from the metal surface influences the high-frequency surface impedance. If the surface were strictly regular the reflection of electrons should be specular. But even a single crystal always has surface irregularities of sizes comparable with interatomic spacings, though they may be small compared with the effective skin depth. The reflection of electrons from the surface will therefore be diffuse or almost diffuse. This means that the distribution function of the electrons reflected from the surface is uncorrelated with that of the incident electrons.

The present paper discusses cyclotron resonance in an inclined magnetic field, and investigates in the case of a parallel field the influence of the reflection coefficient of electrons at the metal surface on the impedance.

2. CYCLOTRON RESONANCE IN AN INCLINED MAGNETIC FIELD

The calculation of the complete surface impedance tensor

$$Z_{\mu\nu} \equiv R_{\mu\nu} + iX_{\mu\nu} = \partial E_\mu(0)/\partial I_\nu = (4\pi\omega/ic^2)\,\partial E_\mu(0)/\partial E'_\nu(0) \qquad (\mu,\ \nu = x,\ y) \tag{2.1}$$

reduces to a simultaneous solution of Maxwell's equations

$$\operatorname{curl}\mathbf{E} = -i\omega\mathbf{H}/c; \qquad \operatorname{div}\mathbf{H} = \operatorname{div}\mathbf{E} = 0; \quad \operatorname{curl}\mathbf{H} = 4\pi\mathbf{j}/c; \qquad \mathbf{j} = -\frac{2e}{h^3}\int \mathbf{v} f_1\, d\mathbf{p} \tag{2.2}$$

and of the kinetic equation for the departure f_1 from an equilibrium Fermi distribution function $f_0(\epsilon)$

$$i\omega f_1 + v_z \frac{\partial f_1}{\partial z} + \Omega \frac{\partial f_1}{\partial \tau} + \frac{f_1}{t_0} = e\mathbf{E}\mathbf{v} \frac{\partial f_0}{\partial \varepsilon}. \tag{2.3}$$

Here $R_{\mu\nu}$ and $X_{\mu\nu}$ are the resistive and reactive parts of the impedance tensor, $E_\mu(z)$ is the tangential component of electric field, I_ν is a component of the total current, ω is the applied frequency, $\Omega = eH/mc$ is the "cyclotron" frequency, $m = (1/2\pi)\partial S(\epsilon, p_H)/\partial\epsilon$, $S(\epsilon, p_H)$ is the area cut out from the surface $\epsilon(\mathbf{p}) = \epsilon$ by the plane $p_H = \text{const}$, τ is the dimensionless time of revolution of an electron in its orbit, $t_0(\mathbf{p})$ is the relaxation time, the z-axis is the inward normal to the metal surface, and the x-axis is the projection of the constant magnetic field $\mathbf{H}$ onto the surface. The boundary coniditon for the kinetic equation (2.3) is in the case of diffuse scattering

$$f_1 = 0 \text{ for } z = 0,\ v_z > 0. \tag{2.4}$$

We have found[2,3] that the surface impedance tensor $Z_{\mu\nu}$ is conveniently deduced from Eq. (2.2) and (2.3) by taking Fourier transforms with respect to z. After calculations precisely analogous to those made earlier[2,3], we obtain the following equations for the Fourier transforms.

$$-k^2\mathscr{E}_\mu(k) - 2E'_\mu(0) = (4\pi i\omega/c^2)\, j_\mu(k); \quad j_\mu(k) = \sum_\nu \left\{K_{\mu\nu}(k)\,\mathscr{E}_\nu(k) - \int_0^\infty Q_{\mu\nu}(k;\ k')\,\mathscr{E}_\nu(k')\,dk'\right\}; \tag{2.5}$$

$$\mathscr{E}_\mu(k) = 2\int_0^\infty E_\mu(z)\cos kz\,dz; \quad j_\mu(k) = 2\int_0^\infty j_\mu(z)\cos kz\,dz,$$

$$K_{\mu\nu}(k) = -\frac{2e^2}{h^3}\int_0^\infty \frac{\partial f_0}{\partial \varepsilon}\,d\varepsilon \int_{p_H^{\min}}^{p_H^{\max}} \frac{m}{\Omega}\,dp_H \cdot \int_0^{2\pi} v_\mu(\tau)\,d\tau \int_{-\infty}^{\tau} v_\nu(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma\,d\tau_2\right)\cos\left(\frac{k}{\Omega}\int_\tau^{\tau_1} v_z\,d\tau_2\right)d\tau_1,$$

$$Q_{\mu\nu}(k;\ k') = -\frac{2e^2}{\pi h^3}\int_0^\infty \frac{\partial f_0}{\partial \varepsilon}\,d\varepsilon \int \frac{m}{\Omega^2}\,dp_H \int_0^{2\pi} v_\mu\,d\tau \int_{-\infty}^{\tau} |v_z(\tau_1)|\cos\left(\frac{k}{\Omega}\int_\tau^{\tau_1} v_z\,d\tau_2\right)d\tau_1 \int_{\varphi(\tau_1)}^{\tau_1} v_\nu(\xi)\exp\left(\int_\tau^{\xi} v_z\,d\tau_2\right)\cos\left(\frac{k'}{\Omega}\int_\xi^{\tau_1} v_z\,d\tau_2\right)d\xi,$$

$$\gamma = i\omega/\Omega + 1/\Omega t_0, \tag{2.6}$$

Here $\varphi(\tau)$ denotes the root immediately preceding τ of the equation

$$\int_{\varphi(\tau)}^{\tau} v_z\,d\tau_2 = 0.$$

If there is no such root, then $\varphi(\tau) = -\infty$. The quantity (kv/Ω), which appears in the argument of the cosines in Eq. (2.6), is of order $(r/\delta_{\text{eff}}) \gg 1$. Therefore the intégrals (2.6) can be evaluated by the method of steepest descent. Elementary but very lengthy calculation then leads to the following results.

To the lowest order in $(\Omega/kv)^{1/2} \sim (\delta_{\text{eff}}/r)^{1/2} \ll 1$, the surface impedance is independent of magnetic field and is equal to its value for $H = 0$. In the first and second higher orders, the dependence on H does not have a resonant behavior. A weakly-resonant dependence appears only in still higher terms in the expansion of the impedance in powers of $(\delta_{\text{eff}}/r)^{1/2}$. Also only the Z_{yy} component of the surface impedance tensor shows a resonance. Thus resonant absorption of power can occur only for electromagnetic waves incident on the metal with a definite polarization. The electric vector of the incident wave must be parallel to the y-axis (perpendicular both to H and to the normal to the surface).

The relative magnitude of the resonant contribution is

$$\frac{\Delta Z_{\text{res}}}{Z(0)} \sim A\left(\frac{\delta_{\text{eff}}}{r}\right)^{3/2}\ln\left[\frac{1}{1 - \exp(-2\pi i\omega/\Omega_0 - 2\pi\nu_0/\Omega_0)}\right], \tag{2.7}$$

At resonance (when $\omega \approx q\Omega_0$ with q an integer) this contribution has only a logarithmic singularity for $(\nu_0/\omega) \ll 1$. Here Ω_0, r, and $\nu_0 = \overline{(1/t_0)}$ are the Larmor frequency, the orbit radius, and the average along the orbit of the collision rate, for an electron with $\bar{v}_H = 0$. A is a complex coefficient independent of H,

$$\delta_{\text{eff}} = |c^2 Z(0)/4\pi\omega| \sim (\delta^2 l)^{1/3}; \quad \delta = (mc^2/2\pi ne^2)^{1/2},$$

and ℓ is the mean free path. The resonant effect is small because it is produced only by the small minority of electrons which return many times into the skin depth. Since the resonant contribution is so small we do not exhibit the exact formulae for it.

The results are correct so long as the angle Φ between the magnetic field and the metal surface is not close to zero or to $\frac{1}{2}\pi$, i.e., so long as $\sin 2\Phi \gg \delta_{\text{eff}}/r$.

The case of a perpendicular magnetic field ($\cos\Phi \ll \delta_{\text{eff}}/r$) is special, because in this case the electrons which give the main contribution to the current density have orbits lying entirely within the skin depth. We have then not cyclotron resonance ($\omega = q\Omega$) but diamagnetic resonance ($\omega = \Omega$) similar to the diamagnetic resonance in semiconductors. Azbel' and Kaganov[5] showed that, to the lowest order in $(\delta_{\text{eff}}/\ell)$ [in a perpendicular field the expansion parameter is $(\delta_{\text{eff}}/\ell)$ instead of (δ_{eff}/r)], the effective mean free path $\ell^* = \ell/[1 + i(\omega - \Omega)t_0]$ in general disappears from the formula for the surface impedance. The impedance is in this approximation independent of H. A resonance (i.e., an extremum with respect to H) appears only in a second-order term proportional to $(\delta_{\text{eff}}/\ell^*)^2 \ln(\ell^*/\delta_{\text{eff}})$.

3. CYCLOTRON RESONANCE IN A PARALLEL MAGNETIC FIELD WITH ARBITRARY REFLECTION LAW OF ELECTRONS AT THE METAL SURFACE

To study the dependence of the surface impedance on the reflection coefficient of electrons at the metal surface, we have to change the boundary condition (2.4). We suppose that a fraction ρ of the electrons is reflected specularly (i.e., without any change in their distribution function), while the rest are distributed after reflection in the equilibrium distribution $f_0(\epsilon)$. This gives the condition

$$f(0;\ v_x,\ v_y,\ v_z) = \rho f(0;\ v_x,\ v_y,\ -v_z) + (1-\rho) f_0(\varepsilon) \text{ for } v_z > 0$$

For the departure from equilibrium $f_1 = f - f_0(\epsilon)$, the condition becomes

$$f_1(0;\ v_x,\ v_y,\ v_z) = \rho f_1(0;\ v_x,\ v_y,\ -v_z), \quad v_z > 0. \tag{3.1}$$

We showed earlier[2] that the Fourier transform of the current density can be determined if we know the relation between $\partial\psi_-(0;\mathbf{v})/\partial z$ and $\psi_-(0;\mathbf{v})$, where $\psi_\pm(z;\mathbf{v}) = f_1(z;\mathbf{v}) \pm f_1(z;-\mathbf{v})$. Equation (2.3) implies

$$\partial\psi_-(z;\ \mathbf{v})/\partial z = -\{i\omega + 1/t_0(\mathbf{p}) + \Omega\partial/\partial\tau\}\psi_+(z;\ \mathbf{v})/v_z \tag{3.2}$$

using also the fact that $\epsilon(\mathbf{p})$ is an even function. We need therefore only find the relation between $\psi_+(0;\mathbf{v})$ and $\psi_-(0;\mathbf{v})$. The boundary condition (3.1) gives

$$\psi_+(0;\ \mathbf{w},\ v_z) = \rho\psi_+(0;\ \mathbf{w},\ -v_z) + \operatorname{sgn} v_z \cdot \{\rho\psi_-(0;\ \mathbf{w};\ -v_z) - \psi_A(0;\ \mathbf{w},\ v_z)\}, \tag{3.3}$$

where $\mathbf{w}$ is the two-dimensional vector with components (v_x, v_y), and $\operatorname{sgn} x$ denotes the sign of x. Changing v_z into $-v_z$ in Eq. (3.3) and eliminating $\psi_+(0;\mathbf{w},-v_z)$, we find

$$\psi_+(0;\ v_x,\ v_y,\ v_z) = -\frac{\operatorname{sgn} v_z}{1-\rho^2}\{(1+\rho^2)\psi_-(0;\ v_x,\ v_y,\ v_z) - 2\rho\psi_-(0;\ v_x;\ v_y,\ -v_z)\}. \tag{3.4}$$

When $\rho \to 0$ we obtain the boundary condition used in Ref. 2. When $\rho \to 1$ we find

$$\psi_-(0;\ \mathbf{w};\ v_z) \to \psi_-(0;\ \mathbf{w};\ -v_z).$$

The Fourier transform of the current density can be found from Eq. (3.4). For simplicity we write down the formula for $\mathbf{j}(\mathbf{k})$ only for the case of an isotropic quadratic dispersion law $\epsilon(\mathbf{p}) = p^2/2m$, where m is the effective mass, assuming a relaxation time t_0 independent of $\mathbf{p}$ (the resistance being in the residual range), and taking the radio-frequency and constant magnetic fields perpendicular to each other. Under these conditions we have

$$j_\mu(k) = K(k)\mathscr{E}_\mu(k) - \int_0^\infty Q(k;\ k')\mathscr{E}_\mu(k')\,dk'; \quad K(k) = \frac{3\sigma\gamma}{\pi}\int_0^{\pi/2}\cos^2\vartheta\sin\vartheta\,d\vartheta \cdot \int_0^\infty e^{-\gamma x}dx \cdot \int_0^\pi \cos[kr\sin\vartheta(\cos\tau - \cos(\tau - x))]\,d\tau;$$

$$Q(k;\ k') = \frac{3\sigma\gamma r}{\pi^2}\int\limits_0^{\pi/2}\cos^2\vartheta\sin^2\vartheta\,d\vartheta\int\limits_0^{\infty}e^{-\gamma x}dx\cdot I(\vartheta;\ x);$$

$$I(\vartheta;\ x) = \int\limits_0^{\pi}d\lambda\cdot\sin\lambda\,\frac{\cos\left[kr\sin\vartheta(\cos\lambda - \cos(\lambda+x))\right] - \rho\cos\left[kr\sin\vartheta(\cos\lambda - \cos(\lambda - x))\right]}{e^{\gamma\lambda} - \rho e^{-\gamma\lambda}}$$

$$\times\int\limits_0^{\lambda}\cosh\gamma\eta\cdot\cos\left[k'r\sin\vartheta(\cos\lambda - \cos\eta)\right]d\eta;\quad \sigma = \frac{ne^2 l}{mv};\quad \gamma = \frac{r}{l};\quad r = \frac{mvc}{eH};\quad l = \frac{vt_0}{1 + i\omega t_0};\quad v = \left(\frac{2\zeta}{m}\right)^{1/2};$$

$$\mathbf{v} = v(\cos\vartheta,\ \sin\vartheta\cos\tau,\ \sin\vartheta\sin\tau);\ n = 8\pi p^3/3h^3. \tag{3.5}$$

When the skin effect is anomalous $kr \sim (r/\delta_{\text{eff}}) \gg 1$. The method of steepest descent then gives the asymptotic formula

$$K(k) = \frac{3\pi\sigma}{4l}\,\frac{1 + \exp(-2\pi\gamma)}{1 - \exp(-2\pi\gamma)}\cdot\frac{1}{k},$$

$$Q(k;\ k') = \frac{3\sigma}{4l}\,\frac{\exp(-2\pi\gamma)}{1 - \exp(-2\pi\gamma)}\,\frac{1-\rho}{1 - \rho\exp(-2\pi\gamma)}\,\frac{1}{\sqrt{kk'}} \tag{3.6}$$

$$\times\left[\pi\delta(k - k') + \frac{\cosh^2\pi\gamma}{k + k'}\right] + \frac{3(1+\rho)\sigma}{4\pi l}\left\{\frac{\cosh\pi\gamma\cdot\exp(-\pi\gamma)}{1 - \rho\exp(-2\pi\gamma)} + \left[1 - \rho + \frac{\sqrt{2}\gamma}{\pi B(3/4,\ 3/2)}\,\frac{\ln(k/k')}{\sqrt{kr} - \sqrt{k'r}}\right]^{-1}\right\}\frac{\ln(k/k')}{k^2 - k'^2},$$

where $B(p, q)$ is the Eulerian integral of the second kind. Equation (3.6) may be regarded as an interpolation formula, since it gives the correct result both for

$$1 - \rho \gg |\gamma\sqrt{\delta_{\text{eff}}/r}|\ \text{and for}\ 1 - \rho \ll |\gamma\sqrt{\delta_{\text{eff}}/r}|.$$

Detailed calculation shows that, for all values of $\rho \neq 1$ [more precisely, for $1 - \rho \gg |\gamma|(\delta_{\text{eff}}/r)^{1/2}$], the surface impedance near to resonance ($\omega \approx q\Omega$) or in the strong field regime ($|\gamma| \ll 1$) is independent of the reflection coefficient ρ. The formulae for Z which we obtained earlier[1-3] remain valid.

The case of specular reflection is special, because in this case the main contribution to the current density comes from electrons which collide repeatedly with the surface. To the lowest order in $(\delta_{\text{eff}}/r)^{1/2}$ the impedance has a non-resonant dependence on magnetic field. Resonance appears only in a higher-order term of order (δ_{eff}/r).

In strong fields ($|\gamma| \ll 1$) the field-dependence of the impedance is quite different for $\rho = 1$ and for $\rho \neq 1$. We find

$$Z(H) \sim Z(0)\gamma^{1/6}(\delta/l)^{2/15} \sim H^{-1/6}\quad(\rho = 1);\qquad Z(H) \sim Z(0)\gamma^{1/3} \sim H^{-1/3}\quad(1 - \rho \gg |\gamma|(\delta_{\text{eff}}/r)^{1/2}). \tag{3.7}$$

The exact formulae for the impedance with $\rho = 1$ will not be reproduced here, since this case is of purely theoretical interest.

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ANGULAR DISTRIBUTION AND POLARIZATION OF ELECTRONS FROM THE BETA-DECAY OF ORIENTED NUCLEI

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The angular distributions and polarizations of electrons (or positrons) from the β-decay of oriented nuclei are calculated for allowed transitions with inclusion of effects of the nonconservation of parity.

RECENTLY conducted experiments[1] have confirmed the hypothesis of Lee and Yang[2] that parity is not conserved in weak interactions. In a paper of Landau[3] the idea has been put forward that the Hamiltonian of the weak interactions is invariant with respect to combined inversion, i.e., with respect to two operations carried out in succession — inversion of space and charge conjugation (despite the possible non-invariance of the Hamiltonian in question with respect to each of these operations individually). The invariance of the theory with respect to combined inversion leads in particular to the reality of all the constants in the interaction of β-decay.

In the present paper we determine for allowed transitions the β-neutrino correlation, the angular distribution, and the polarization of the electrons (or positrons) produced in the β-decay of oriented nuclei. In doing this we make no preliminary assumptions about the real or imaginary nature of the different interaction constants. Therefore the formulas obtained may turn out to be useful for clearing up the question of the existence of invariance with respect to the combined inversion.

The most general expression for the Hamiltonian of the β-interaction in the case of nonconservation of parity has been given in Ref. 2. It has the form

$$\begin{gathered} H = (\psi_p^+\gamma_4\psi_n)(C_S\psi_e^+\gamma_4\psi_\nu + C_S'\psi_e^+\gamma_4\gamma_5\psi_\nu) \\ + (\psi_p^+\gamma_4\gamma_\mu\psi_n)(C_V\psi_e^+\gamma_4\gamma_\mu\psi_\nu + C_V'\psi_e^+\gamma_4\gamma_\mu\gamma_5\psi_\nu) + {}^1/_2(\psi_p^+\gamma_4\sigma_{\lambda\mu}\psi_n)(C_T\psi_e^+\gamma_4\sigma_{\lambda\mu}\psi_\nu + C_T'\psi_e^+\gamma_4\sigma_{\lambda\mu}\gamma_5\psi_\nu) \\ + (\psi_p^+\gamma_4\gamma_\mu\gamma_5\psi_n)(-C_A\psi_e^+\gamma_4\gamma_\mu\gamma_5\psi_\nu - C_A'\psi_e^+\gamma_4\gamma_\mu\psi_\nu) + (\psi_p^+\gamma_4\gamma_5\psi_n)(C_P\psi_e^+\gamma_4\gamma_5\psi_\nu + C_P'\psi_e^+\gamma_4\psi_\nu) + \text{Hermitian conj.} \end{gathered} \tag{1}$$

Since in the following we consider only allowed transitions, in the calculation of the matrix elements we can neglect the pseudoscalar interaction and replace the wave functions of the electron (or positron) and antineutrino (or neutrino) by their values for $r \to 0$.

For definiteness let us first consider electron emission. The wave function of an electron moving in the Coulomb field of the nucleus and having at infinity the momentum $\mathbf{p}$ and the polarization μ is described by the wave function

$$\begin{gathered} \psi_{\mathbf{p}\mu} = 8\pi\sqrt{\frac{m}{2\varepsilon}}\sum_{jlM}(v_\mu,\ \Omega^*_{jlM}(\mathbf{p}))\, i^{l-1}e^{-i\delta_{jl}}\,\frac{e^{\pi\nu/2}\,|\Gamma(\gamma_{jl}+i\nu)|}{\Gamma(2\gamma_{jl}+1)}(2pr)^{\gamma_{jl}-1} \\ \times\begin{pmatrix} i\sqrt{\frac{\varepsilon}{m}+1}\,\mathrm{Re}\left[e^{-i(pr-\eta_{jl})}(\gamma_{jl}+i\nu)F(\gamma_{jl}+1+i\nu,\,2\gamma_{jl}+1;2ipr)\right]\Omega_{jlM}(\mathbf{r}) \\ \sqrt{\frac{\varepsilon}{m}-1}\,\mathrm{Im}\left[e^{-i(pr-\eta_{jl})}(\gamma_{jl}+i\nu)F(\gamma_{jl}+1+i\nu,\,2\gamma_{jl}+1;\,2ipr)\right]\Omega_{jl'M}(\mathbf{r}) \end{pmatrix} \end{gathered} \tag{2}$$

Here we have used the notations adopted in the book of Akhiezer and Berestetskii[4] [in this usage v_μ is a normalized two-component spinor, $(v_\mu^*, v_\mu) = 1$]. At infinity the function $\psi_{p\mu}$ has the form of the superposition of a plane wave and a converging spherical wave and is normalized to unit amplitude of the plane wave. For $r \to 0$ one keeps in the sum over j and ℓ in Eq. (2) only the two terms with $j = \frac{1}{2}$ and $\ell = 0, 1$, and sets $r = 0$ everywhere except in the factor $(2pr)^{\gamma_0 - 1}$, where $\gamma_0 = (1 - Z^2e^2)^{1/2}$. Then we get

$$\psi_{\substack{p\mu \\ r\to 0}} = \frac{\Gamma(\gamma_0+i\nu)\, e^{\pi\nu/2} e^{i\pi(\gamma_0-1)/2}}{\sqrt{2}\,\Gamma(2\gamma_0+1)} (2pr)^{\gamma_0-1} \left(\begin{array}{l} \Big\{\Big[(1+\gamma_0)\sqrt{1+\frac{m}{\varepsilon}} + iZe^2\sqrt{1-\frac{m}{\varepsilon}}\Big] + \\ \qquad + \Big[(\gamma_0-1)\sqrt{1+\frac{m}{\varepsilon}} + iZe^2\sqrt{1-\frac{m}{\varepsilon}}\Big](\sigma\mathbf{n}_r)(\sigma\mathbf{n}_p)\Big\} v_\mu(\xi) \\ \Big\{\Big[(\gamma_0-1)\sqrt{1-\frac{m}{\varepsilon}} + iZe^2\sqrt{1+\frac{m}{\varepsilon}}\Big](\sigma\mathbf{n}_r) + \\ \qquad + \Big[(\gamma_0+1)\sqrt{1-\frac{m}{\varepsilon}} + iZe^2\sqrt{1+\frac{m}{\varepsilon}}\Big](\sigma\mathbf{n}_p)\Big\} v_\mu(\xi), \end{array} \right), \tag{3}$$

where $\mathbf{n}_r$ and $\mathbf{n}_p$ are unit vectors in the directions of **r** and **p**. In allowed transitions there is no change of the parity of the nuclear state, and therefore those terms in Eq. (3) which contain the factor $\mathbf{n}_r$ contribute nothing to the matrix elements of the transitions in question. Dropping them and making the usual replacement of $(2pr)^{\gamma_0-1}$ by $(2pR)^{\gamma_0-1}$ (R is the radius of the nucleus), we can rewrite Eq. (3) in the following way:

$$\psi^0_{p\mu} = f u^c_{p\mu}(\xi), \tag{4}$$

where

$$f = \frac{\sqrt{2(1+\gamma_0)}\,\Gamma(\gamma_0+i\nu)\, e^{\pi\nu/2} e^{i\pi(\gamma_0-1)/2}}{\Gamma(2\gamma_0+1)} (2pR)^{\gamma_0-1}, \tag{5}$$

and $u^c_{p\mu}(\xi)$ is a spin function (ξ is the spin variable) which has the form

$$u^c_{p\mu}(\xi) = \left|\begin{array}{c} a v_\mu \\ b(\sigma\mathbf{n}_p)\cdot v_\mu \end{array}\right|, \tag{6}$$

$$a = \frac{1}{2\sqrt{1+\gamma_0}}\Big[(\gamma_0+1)\sqrt{1+\frac{m}{\varepsilon}} + iZe^2\sqrt{1-\frac{m}{\varepsilon}}\Big], \quad b = \frac{1}{2\sqrt{1+\gamma_0}}\Big[(\gamma_0-1)\sqrt{1-\frac{m}{\varepsilon}} + iZe^2\sqrt{1+\frac{m}{\varepsilon}}\Big]. \tag{7}$$

For the wave function of a free electron with momentum **p** and polarization μ we have at $r = 0$ the expression

$$u^0_{p\mu}(\xi) = \left|\begin{array}{c} a_0 v_\mu \\ b_0(\sigma\mathbf{n}_p) v_\mu \end{array}\right|, \tag{8}$$

$$a_0 = \sqrt{{}^1\!/_2(1+m/\varepsilon)}, \quad b_0 = \sqrt{{}^1\!/_2(1-m/\varepsilon)}. \tag{9}$$

In analogy with Eq. (8), the function ψ_ν corresponding to the production of an antineutrino with momentum **q** (and mass zero) has at $r = 0$ the form

$$u_{q\nu} = \frac{1}{\sqrt{2}}\left|\begin{array}{c} v_\nu \\ ((\sigma\mathbf{q})/q) v_\nu \end{array}\right|. \tag{10}$$

The β-decay matrix element in which we are interested can now be written in the following way:

$$M = f\,(u^{c*}_{p\mu} O u_{q\nu}), \tag{11}$$

where

$$O = A[\gamma_4(C_S + C'_S\gamma_5) + (C_v + C'_v\gamma_5)] - i\mathbf{B}[\gamma\,(C_T\gamma_5 + C'_T) + \gamma_4\gamma\,(C_A\gamma_5 + C'_A)], \tag{12}$$

$$A = \int \Psi_f^* \Psi_i\, d\tau, \quad \mathbf{B} = \int \Psi_f^* \sigma \Psi_i\, d\tau, \tag{13}$$

and Ψ_i and Ψ_f are the initial and final wave functions of the nucleus.

To calculate the probability of the decay we must sum the square of the absolute value of the expression (11) over the spins of the electron and antineutrino. This summation is easily carried out by means of the matrices N and P,

$$N_{\eta\eta'} = \sum_\nu u_{q\nu}(\eta)\, u^*_{q\nu}(\eta'), \quad P_{\xi\xi'} = \sum_\mu u^c_{p\mu}(\xi)\, u^{c*}_{p\mu}(\xi') \tag{14}$$

and gives

$$\sum_{\mu\nu} |M|^2 = |f|^2 \sum_{\mu\nu} (u^{c*}_{p\mu} O u_{q\nu}) (u^{*}_{q\nu} O^{+} u^{c}_{p\mu}) = |f|^2 \operatorname{Sp}(ONO^{+}P); \tag{15}$$

N is the usual projection operator

$$N(\mathbf{q}) = -i\hat{q}\gamma_4 / 2q. \tag{16}$$

By means of Eqs. (6) and (7) it is not hard to obtain also the explicit expression for the matrix P:

$$P = \begin{vmatrix} |a|^2 & ab^{*}\sigma\mathbf{n}_p \\ a^{*}b\sigma\mathbf{n}_p & |b|^2 \end{vmatrix} = \frac{m - i\hat{p}}{2\varepsilon}\gamma_4 + \frac{m}{2\varepsilon}[(\gamma_0 - 1) + Ze^2\mathbf{n}_p\gamma]. \tag{17}$$

We finally get for the differential probability of the β-decay (in units $mc^2/\hbar$) the expression

$$N\left(\mathbf{p}, \frac{\mathbf{q}}{q}\right)\frac{d\varepsilon}{m} d\Omega_e d\Omega_\nu = F\left(Z, \frac{\varepsilon}{m}\right)\overline{\operatorname{Sp}(ONO^{+}P)}\,(2\pi m)^{-5} p\varepsilon(\varepsilon_0 - \varepsilon)^2 \, d\varepsilon d\Omega_e d\Omega_\nu, \tag{18}$$

where the bar over the factor denotes averaging with respect to M and summing over M′, these being the magnetic quantum numbers of the initial and final states of the nucleus, respectively; ϵ_0 is the maximum energy of the electrons; and $F(Z, \epsilon/m)$ is the usual notation for the function which includes the influence of the Coulomb field of the nucleus:[5]

$$F(Z, \varepsilon/m) = |f|^2. \tag{19}$$

To calculate the polarization of the emitted electrons we introduce the density matrix

$$\rho(\xi, \xi') = Q \overline{\sum_{\mu\lambda\mu} u^{0}_{\mu}(\xi) (u^{c*}_{\mu}(\eta) O_{\eta\eta'} u_{\nu}(\eta')) (u^{*}_{\nu}(\zeta) O^{+}_{\zeta\zeta'} u^{c}_{\lambda}(\zeta')) u^{0*}_{\lambda}(\xi')}. \tag{20}$$

Q is a constant to be determined from the normalization condition $\operatorname{Sp}\rho = 1$. We further define the matrix R:

$$R_{\xi\eta} = \sum_{\mu} u^{0}_{\mu}(\xi) u^{c*}_{\mu}(\eta). \tag{21}$$

By the use of Eqs. (6) — (9) we find

$$R = \sqrt{\frac{\gamma_0 + 1}{2}} \frac{(m - i\hat{p})\gamma_4}{2\varepsilon}\left[1 - i\frac{Ze^2}{(\gamma_0 + 1)p}(\varepsilon - m\gamma_4)\right]. \tag{22}$$

It is not hard to verify that $R^{+}R = P$, and therefore the correctly normalized matrix ρ can be written in the following form:

$$\rho = RONO^{+}R^{+} / \overline{\operatorname{Sp}(ONO^{+}P)}. \tag{23}$$

If we do not prescribe the direction of emission of the neutrino, then we must proceed further to average the numerator and the denominator in Eq. (23) over the direction of **q**. We then get for the density matrix

$$\rho = \int (RONO^{+}R^{+})\, d\Omega_\nu \Big/ \int \overline{\operatorname{Sp}(ONO^{+}P)}\, d\Omega_\nu. \tag{24}$$

On the other hand, the general expression for the density matrix ρ describing the polarization properties of the electrons moving with the momentum **p** has the form[6]

$$\rho = (1 + i\gamma_5\hat{\zeta})(m - i\hat{p})\gamma_4 / 4\varepsilon. \tag{25}$$

The four-vector $\zeta_\mu = (\zeta, \zeta_0)$ satisfies the condition $\zeta_\mu p_\mu = 0$. Therefore in a system of coordinates in which $\mathbf{p} = 0$, we have $\zeta_0 = 0$, i.e., $\zeta^0_\mu = (\zeta^0, 0)$. Consequently, the polarization properties can be defined, as is usually done, by means of the three-dimensional vector ζ^0 connected with ζ by the relation

$$\zeta^0 = \zeta_t + (m/\varepsilon)\zeta_l, \tag{26}$$

where ζ_t and ζ_l are the transverse and longitudinal components of the vector ζ. Equations (24) and (25) give

$$\zeta_\mu = \frac{i\varepsilon}{m}\,\mathrm{Sp}\,(\rho\gamma_4\gamma_5\gamma_\mu) = \frac{i\varepsilon}{m}\int \mathrm{Sp}\,\overline{(RONO^+R^+\gamma_4\gamma_5\gamma_\mu)}\,d\Omega_\nu \Big/ \int \overline{\mathrm{Sp}\,(ONO^+P)}\,d\Omega_\nu. \tag{27}$$

The further calculation of the angular distributions by means of Eqs. (12), (13), and (16) – (18) and of the polarization of the electrons by means of these same equations and Eqs. (26) and (27) is carried out in the usual way. Therefore we at once give below the results of these calculations for the three types of allowed transitions, $J' = J,\ J \pm 1$ for both the electron and the positron decays.* Here we have introduced, in addition to the usually accepted notations,[2,5] also the following: $\mathbf{n}_\nu = \mathbf{q}/q$, and $\mathbf{j}$ is the unit vector in the direction of orientation of the nuclei. In what follows the index zero is omitted from the quantity $\gamma_0 = (1 - Z^2e^2)^{1/2}$.

I. Correlation of directions of the β particle and the neutrino

$$N(\mathbf{p}, \mathbf{n}_\nu)\,dW\,d\Omega_e\,d\Omega_\nu = (2\pi)^{-5}\xi F\,(\pm Z, W)\,(W^2-1)^{1/2}\,W\,(W_0-W)^2 dW\,d\Omega_e d\Omega_\nu$$

$$\times \Big\{1 + \Big(a_1 + \frac{Ze^2}{Wv_e}a_2\Big)(\mathbf{v}_e\mathbf{n}_\nu) + \frac{b}{W} + \frac{\langle J_z\rangle}{J}\Big[\Big(c_1 + \frac{Ze^2}{Wv_e}c_2\Big)(\mathbf{j}\mathbf{v}_e)$$

$$+ \Big(d_1 + \frac{\gamma}{W}d_2\Big)(\mathbf{j}\mathbf{n}_\nu) + \Big(g_1 + g_2\frac{Ze^2}{Wv_e}\Big)(\mathbf{j}\,[\mathbf{n}_\nu\times\mathbf{v}_e])\Big] + \Big(f_1 + \frac{Ze^2}{Wv_e}f_2\Big)\Big(\frac{1}{3}\,\mathbf{v}_e\mathbf{n}_\nu - (\mathbf{v}_e\mathbf{j})\,(\mathbf{n}_\nu\mathbf{j})\Big)\Big(1 - \frac{3\langle J_z^2\rangle}{J(J+1)}\Big)\Big\}. \tag{28}$$

Here

$$\xi = (|C_T|^2 + |C_T'|^2 + |C_A|^2 + |C_A'|^2)\,|M_{\mathrm{GT}}|^2 + (|C_S|^2 + |C_S'|^2 + |C_V|^2 + |C_V'|^2)\,|M_{\mathrm{F}}|^2,$$

$$a_1\xi = \frac{1}{3}(|C_T|^2 + |C_T'|^2 - |C_A|^2 - |C_A'|^2)\,|M_{\mathrm{GT}}|^2 - (|C_S|^2 + |C_S'|^2 - |C_V|^2 - |C_V'|^2)\,|M_{\mathrm{F}}|^2,$$

$$a_2\xi = \pm[-\frac{2}{3}\,\mathrm{Im}\,(C_AC_T^* + C_A'C_T'^*)\,|M_{\mathrm{GT}}|^2 + 2\,\mathrm{Im}\,(C_VC_S^* + C_V'C_S'^*)\,|M_{\mathrm{F}}|^2],$$

$$b\xi \doteq \pm 2\gamma\,[\mathrm{Re}\,(C_TC_A^* + C_T'C_A'^*)\,|M_{\mathrm{GT}}|^2 + \mathrm{Re}\,(C_SC_V^* + C_S'C_V'^*)\,|M_{\mathrm{F}}|^2],$$

$$c_1\xi = \pm 2\,\mathrm{Re}\,(C_TC_T'^* - C_AC_A'^*)\,|M_{\mathrm{GT}}|^2\varepsilon_{JJ'} + 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[(C_TC_S'^* + C_T'C_S^* - C_AC_V'^* - C_A'C_V^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}],$$

$$c_2\xi = 2\,\mathrm{Im}\,(C_TC_A'^* + C_T'C_A^*)\,|M_{\mathrm{GT}}|^2\varepsilon_{JJ'} \pm 2\sqrt{\frac{J}{J+1}}\,\mathrm{Im}\,[(C_TC_V'^* + C_T'C_V^* - C_AC_S'^* - C_A'C_S^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}],$$

$$d_1\xi = \pm 2\,\mathrm{Re}\,(C_TC_T'^* + C_AC_A'^*)\,|M_{\mathrm{GT}}|^2\varepsilon_{JJ'} - 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[C_TC_S'^* + C_T'C_S^* + C_AC_V'^* + C_A'C_V^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}],$$

$$d_2\xi = \pm 2\,\mathrm{Re}\,(C_TC_A'^* + C_T'C_A^*)\,|M_{\mathrm{GT}}|^2\varepsilon_{JJ'} - 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[(C_TC_V'^* + C_T'C_V^* + C_AC_S'^* + C_A'C_S^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}],$$

$$g_1\xi = 2\sqrt{\frac{J}{J+1}}\,\mathrm{Im}\,[(C_TC_S^* + C_T'C_S'^* - C_AC_V^* - C_A'C_V'^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}],$$

$$g_2\xi = \mp 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[(C_TC_V^* + C_T'C_V'^* - C_AC_S^* - C_A'C_S'^*)\,M_{\mathrm{F}}^*M_{\mathrm{GT}}].$$

$$f_1\xi = (|C_T|^2 + |C_T'|^2 - |C_A|^2 - |C_A'|^2)\,|M_{\mathrm{GT}}|^2\Delta_{JJ'},\quad f_2\xi = \mp 2\,\mathrm{Im}\,(C_AC_T^* + C_A'C_T'^*)\,|M_{\mathrm{GT}}|^2\Delta_{JJ'}.$$

The upper signs refer everywhere to the electron decay, and the lower signs to the positron decay. The functions $\epsilon_{JJ'}$ and $\Delta_{JJ'}$ are defined by the relations

$$\varepsilon_{JJ'} = \begin{cases} \dfrac{1}{J+1} & \text{if}\quad J' = J, \\ 1, & \text{if}\quad J' = J-1, \\ -\dfrac{J}{J+1}, & \text{if}\quad J' = J+1, \end{cases}\quad \Delta_{JJ'} = \begin{cases} -1, & \text{if}\quad J' = J, \\ \dfrac{J+1}{2J-1}, & \text{if}\quad J' = J-1, \\ \dfrac{J}{2J+3}, & \text{if}\quad J' = J+1. \end{cases} \tag{29}$$

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*The formulas for the positron decay are obtained from the corresponding formulas for the electron decay if one makes in the latter the following replacements: (1) C_S, C_A, C_P, C_V', C_T' by C_S^*, C_A^*, C_P^*, $C_V'^*$, $C_T'^*$; (2) C_S', C_A', C_P', C_V, C_T by $-C_S'^*$, $-C_A'^*$, $-C_P'^*$, $-C_V^*$, $-C_T^*$; and (3) e^2 by $-e^2$.

For unoriented nuclei $\langle J'_Z \rangle = 0$ and $1 - \frac{3\langle J_Z^2 \rangle}{J(J+1)} = 0$.

In virtue of the invariance of the Hamiltonian for the strong interactions with respect to time reversal, the product $M_F^* M_{GT}$ is real,[7] i.e.,

$$M_F^* \cdot M_{GT} = \pm |M_F| \cdot |M_{GT}|. \tag{30}$$

It must be pointed out that the term $(Ze^2/Wv_e)\,a_2(\mathbf{v}_e\mathbf{n}_\nu)$ in the curly brackets of Eq. (28) is omitted through an error from Eq. (A.2) of Lee and Yang's paper.[2]

II. Integration of Eq. (28) over all angles of emission of the neutrino gives the angular distribution of the electrons (or positrons)

$$N(\mathrm{p})\,dW\,d\Omega_e = \frac{\xi}{8\pi^4}\left(1 + \frac{b}{W}\right) F(\pm Z,\ W)\,(W^2 - 1)^{1/2}\,W\,(W_0 - W)^2 dW \cdot (1 + \alpha \cos\vartheta), \tag{31}$$

ϑ is the angle between $\mathbf{v}_e$ and $\mathbf{j}$, and

$$\alpha = \frac{\langle J_z \rangle}{J}\,\frac{c_1 + c_2 Ze^2/Wv_e}{1 + b/W}\,v_e. \tag{32}$$

For the transitions $J \to J - 1$ and $J \to J + J$ Eq. (32) gives results identical with those obtained in Eqs. (A.6) and (A.7) of Ref. 2.

III. The general expression for the polarization of the electrons (or positrons) of energy W emitted at the angle ϑ with the direction of orientation of the nuclei has the following form:

$$\begin{aligned}\zeta^0 = {}&\left\{\xi\left[1 + \frac{b}{W} + \frac{\langle J_z \rangle}{J}\left(c_1 + c_2\frac{Ze^2}{Wv_e}\right) v_e \cos\vartheta\right]\right\}^{-1}\left\{\left(\alpha_1 + \alpha_2\frac{Ze^2}{Wv_e}\right.\right.\\ &\left.+ \alpha_3\,\frac{1 - (\gamma/W)}{v_e}\frac{\langle J_z \rangle}{J}\cos\vartheta\right)\mathbf{v}_e + \left(\beta_1 + \frac{\beta_2}{W}\right)\frac{\langle J_z \rangle}{J}\,\mathbf{j} + \left.\left(\gamma_1 + \gamma_2\frac{Ze^2}{Wv_e}\right)\frac{\langle J_z \rangle}{J}[\mathbf{j},\ \mathbf{v}_e]\right\},\end{aligned} \tag{33}$$

$$\alpha_1 = \pm 2\,[\mathrm{Re}\,(C_T C_T'^* - C_A C_A'^*)\,|M_{GT}|^2 + \mathrm{Re}\,(C_S C_S'^* - C_V C_V'^*)\,|M_F|^2],$$

$$\alpha_2 = 2\,[\mathrm{Im}\,(C_T C_A'^* + C_T' C_A^*)\,|M_{GT}|^2 + \mathrm{Im}\,(C_S C_V'^* + C_S' C_V^*)\,|M_F|^2],$$

$$\begin{aligned}\alpha_3 = {}&[|C_T|^2 + |C_T'|^2 + |C_A|^2 + |C_A'|^2 \mp 2\,\mathrm{Re}\,(C_T C_A^* + C_T' C_A'^*)]\,|M_{GT}|^2 \varepsilon_{JJ'} + 2\sqrt{\frac{J}{J+1}}\{\pm\,\mathrm{Re}\,[(C_T C_S^*\\ &+ C_T' C_S'^* + C_A' C_V'^* + C_A C_V^*)\,M_F^* M_{GT}] - \mathrm{Re}\,[(C_T C_V^* + C_T' C_V'^* + C_A C_S^* + C_A' C_S'^*)\,M_F^* M_{GT}]\},\end{aligned}$$

$$\beta_1 = \pm 2\,\mathrm{Re}\,(C_T C_A^* + C_T' C_A'^*)\,|M_{GT}|^2 \varepsilon_{JJ'} + 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[(C_T C_V^* + C_T' C_V'^* + C_A C_S^* + C_A' C_S'^*)\,M_F^* M_{GT}],$$

$$\beta_2 = \gamma\left\{(|C_T|^2 + |C_T'|^2 + |C_A|^2 + |C_A'|^2)\,|M_{GT}|^2 \varepsilon_{JJ'} \pm 2\sqrt{\frac{J}{J+1}}\;\mathrm{Re}\,[(C_T C_S^* + C_T' C_S'^* + C_A' C_V'^* + C_A C_V^*)\,M_F^* M_{GT}]\right\},$$

$$\gamma_1 = \pm 2\,\mathrm{Im}\,(C_T C_A'^* + C_T' C_A^*)\,|M_{GT}|^2 \varepsilon_{JJ'} + 2\sqrt{\frac{J}{J+1}}\,\mathrm{Im}\,[(C_T C_V'^* + C_T' C_V^* - C_A C_S'^* - C_A' C_S^*)\,M_F^* M_{GT}],$$

$$\gamma_2 = -2\,\mathrm{Re}\,(C_T C_T'^* - C_A C_A'^*)\,|M_{GT}|^2 \varepsilon_{JJ'} \mp 2\sqrt{\frac{J}{J+1}}\,\mathrm{Re}\,[(C_T C_S'^* + C_T' C_S^* - C_A C_V'^* - C_A' C_V^*)\,M_F^* M_{GT}].$$

If the neutrino is a longitudinal particle,[8,9] then we should put $C' = -C$ in all the formulas.

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CONNECTION BETWEEN THE HULTHÉN AND KOHN METHODS IN COLLISION THEORY

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Various possible direct variational methods for the determination of the phase shifts of the radial wave function are considered. It is shown that the most natural criterion of the quality of the trial function is the condition of the consistency of the equations. The comparison of the phase-shift results obtained by the Hulthén and Kohn methods, and the verification of whether the integral identity is satisfied, are not independent criteria and in fact reduce to the consistency condition. It is also shown how the correct value can be chosen from two phase-shift values obtained by the Hulthén method, without resorting to comparison with the results of other methods.

WE consider the equation for the phase shift in collision theory:

$$\psi''(r) + (k^2 - V)\psi(r) = 0; \tag{1}$$

$$\psi(0) = 0, \quad \psi|_{r\to\infty} \sim A\sin(kr + \eta). \tag{2}$$

The variational principle for this problem can be written in the form[1]

$$\delta J = \delta\int_0^\infty \psi(r)\left(\frac{d^2}{dr^2} + k^2 - V\right)\psi(r)\,dr = -A^2k\delta\eta. \tag{3}$$

If we substitute in this functional an approximate function $\widetilde{\psi}(r)$ which satisfies the conditions (2) and which depends on the parameters α_i, we can obtain equations for these parameters from the variational principle. It is well known that, in contrast to the problem of the discrete spectrum, the set of equations can be formed in this case, in a non-unique fashion.

We consider the very simple but also very important case in which a linear combination of n functions $\varphi_i(r)$ are substituted in the functional J:

$$\widetilde{\psi}(r) = \sum_{i=1}^{n} \alpha_i\varphi_i(r). \tag{4}$$

In this case, let

$$\varphi_i(0) = 0, \quad i = 1, 2, \ldots, n;$$

$$\text{for } r \to \infty \quad \varphi_1 \sim \sin kr, \quad \varphi_2 \sim \cos kr, \quad \varphi_j \to 0, \quad j = 3, 4, \ldots, n. \tag{5}$$

Then the functional J will be quadratic in

$$J(\widetilde{\psi}) = \sum_{i,\, j=1}^{n} \alpha_i \alpha_j J_{ij}, \tag{6}$$

where

$$J_{ij} = \frac{1}{2}\left[\int_0^\infty \varphi_i \left(\frac{d^2}{dr^2} + k^2 - V\right)\varphi_j\, dr + \int_0^\infty \varphi_j \left(\frac{d^2}{dr^2} + k^2 - V\right)\varphi_i\, dr\right]. \tag{7}$$

The amplitude and phase for such a choice of wave function are determined by the coefficients α_1 and α_2. It is evident that $\alpha_1 = A\cos\eta$, $\alpha_2 = A\sin\eta$, and

$$A^2\delta\eta = \alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1. \tag{8}$$

Thus, the variational principle (3) can be written in the form

$$\delta J = k(\alpha_2\delta\alpha_1 - \alpha_1\delta\alpha_2). \tag{9}$$

The equations for the coefficients $\alpha_1, \ldots, \alpha_n$ have in this case the form[2]

$$\sum_{j=1}^{n} J_{1j}\alpha_j = \frac{k\alpha_2}{2}; \quad \sum_{j=1}^{n} J_{2j}\alpha_j = -\frac{k\alpha_1}{2}; \quad \sum_{j=1}^{n} J_{ij}\alpha_j = 0 \quad (i = 3, 4, \ldots, n). \tag{10}$$

The condition for the existence of a non-trivial solution of this system,

$$\begin{vmatrix} J_{11} & J_{12} - k/2 & \ldots & J_{1n} \\ J_{21} + k/2 & J_{22} & \ldots & J_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ J_{n1} & J_{n2} & & J_{nn} \end{vmatrix} = 0, \tag{11}$$

is generally not satisfied. Discarding one of the equations (10), we can make the system solvable and arrive at a different formulation of the variational principle. Thus, for example, by discarding the first of these equations, we obtain Kohn's method.[3] The coefficient α_1 is not varied here and is assumed to be equal to unity. Then $\alpha_2 = \tan\eta$, $\delta\alpha_1 = 0$, and

$$\delta J = -k\delta\alpha_2 = -k\delta(\tan\eta). \tag{12}$$

Hulthén[1] discards the first two equations of the system (10) but adds the equation J = 0, which is obtained if each of Eqs. (10) is multiplied by the corresponding α_i and all are added together. The condition J = 0 holds because the functional is stationary relative to a variation of the normalizing constant A.

In the Kohn method[3] for the determination of the coefficients α, we have the set of equations

$$\begin{gathered} J_{21} + J_{22}\alpha_2 + \ldots + J_{2n}\alpha_n = -k/2, \\ J_{31} + J_{32}\alpha_2 + \ldots + J_{3n}\alpha_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ J_{n1} + J_{n2}\alpha_2 + \ldots + J_{nn}\alpha_n = 0. \end{gathered} \tag{13}$$

Solving this set, we find the approximate wave function $\widetilde{\psi}$. Inasmuch as the coefficient α_1 is fixed, we can no longer vary the normalizing constant A freely; consequently, the value of $J(\widetilde{\psi})$ will generally differ from zero. This makes it possible to improve the value for $\tan\eta$ by making use of Eq. (12). Thus,

$$\tan\eta = \alpha_2 + J(\widetilde{\psi})/k. \tag{14}$$

This formula can be transformed by using the system (13). Actually,

$$\begin{aligned} J(\widetilde{\psi}) = \sum_{i=1}^{n} \alpha_i (J_{i1}\alpha_1 + J_{i2}\alpha_2 + \ldots + J_{in}\alpha_n) = \alpha_1 (J_{11}\alpha_1 + J_{12}\alpha_2 + \ldots + J_{1n}\alpha_n) \\ - k\alpha_2/2 = J_{11} + J_{12}\alpha_2 + J_{13}\alpha_3 + \ldots + J_{1n}\alpha_n - k\alpha_2/2, \end{aligned} \tag{15}$$

and Eq. (14) takes the form

$$k \tan \eta = J_{11} + (J_{12} + k/2)\,\alpha_2 + J_{13}\alpha_3 + \ldots + J_{1n}\alpha_n. \tag{16}$$

In order to establish the connection between the methods of Hulthén and Kohn, we eliminate α_3, $\alpha_4, \ldots \alpha_n$ with the aid of the last $(n-2)$ equations of (13). This leads to a system of two equations for α_1 and α_2:

$$A_{11}\alpha_1 + A_{12}\alpha_2 = k\alpha_2/2,\ A_{21}\alpha_1 + A_{22}\alpha_2 = -k\alpha_1/2. \tag{17}$$

Multiplying the first equation by α_1 and the second by α_2, and adding them together, we get the Hulthén equation

$$J = A_{11}\alpha_1^2 + 2A_{12}\alpha_1\alpha_2 + A_{22}\alpha_2^2 = 0. \tag{18}$$

Then, for the desired quantity $\alpha_2/\alpha_1 = \tan\eta$ we get, by Hulthén's method,

$$\tan \eta = (-A_{12} \pm \sqrt{A_{12}^2 - A_{11}A_{22}})/A_{22}. \tag{19}$$

Similarly, by Kohn's method,

$$\alpha_2 = -\frac{1}{A_{22}}\left(A_{12} + \frac{k}{2}\right); \quad \tan \eta = -\frac{1}{A_{22}}\left(A_{12} + \frac{k}{2}\right) + \frac{1}{A_{22}k}\left(A_{11}A_{22} - A_{12}^2 + \frac{k^2}{4}\right). \tag{20}$$

The results obtained by both methods will be identical if Eqs. (17) are consistent, i.e., if the condition

$$A_{12}^2 - A_{11}A_{22} = k^2/4. \tag{21}$$

is satisfied. This condition clearly coincides with condition (11).

It is easy to see that if the radical in (19) is taken with the minus sign, then Eqs. (19) and (20) actually coincide when Eq. (21) is satisfied. In this way, in order to select the true root from the two possible roots of the Hulthén equation, there is no necessity (as is pointed out in Ref. 4, for example) of comparing the results obtained here with the results of two approximate calculations. The root with the minus sign in front of the radical has physical meaning. Furthermore, the condition (21) itself gives a sufficiently reliable criterion of the quality of the wave function thus obtained. It is also evident from this why the Hulthén method does not give complex values for $\tan\eta$, provided the function is chosen satisfactorily. In fact, the radicand is close to the value $k^2/4$ in this case, and is thus positive.

To understand the formal nature of the second solution in the Hulthén method, we note that if condition (21) is satisfied, the second root becomes

$$\tan \eta = (-A_{12} + k/2)/A_{22}. \tag{22}$$

We can get the same value for $\tan\eta$ from a formal attempt to satisfy, in the space of the functions (4), the variational equation

$$\delta J = -k(\alpha_2\,\delta\alpha_1 - \alpha_1\,\delta\alpha_2) = A^2 k\delta\eta, \tag{23}$$

which is obtained from the variational principle (9) by replacement of k by $-k$ on the right hand side.

However, it is evident that Eq. (23) makes no sense. In fact, the wave function is already uniquely determined because the functional J is stationary relative to variations that vanish at infinity. It therefore follows that this result, which is obtained for δJ in the variation of a function of asymptotic form, is unique and cannot be changed at will. Consequently, Eq. (23) in general defines no function at all. If we solve the problem by the Hulthén method, increasing gradually the number n of the functions φ_i, then the second root [the plus sign in Eq. (19)] either will not tend to a definite limit, or this limit will depend essentially on the choice of the sequence of functions φ_i. Actual calculations verify this statement.

Now let us consider in what measure the integral identity, which stems from the variational principle, must be satisfied for the resultant approximate functions. In the given case, this identity has the form

$$\tan \eta = \frac{1}{k}\int_0^\infty \psi(r)\left(\frac{d^2}{dr^2} + k^2 - V\right)\sin kr\,dr = -\frac{1}{k}\int_0^\infty \psi(r)V\sin kr\,dr, \quad \psi|_{r\to\infty} \sim \sin kr + \tan\eta\cos kr. \tag{24}$$

Let us assume that $\varphi_i(r) = \sin kr$; such a choice is simplest; in practice they all behave in this fashion. We substitute the approximate expression (4) in Eq. (24) in place of the exact wave function. We then get

$$\tan\eta = \frac{1}{k}\sum_{i=1}^{n}\alpha_i\int_0^\infty \varphi_i\left(\frac{d^2}{dr^2}+k^2-V\right)\varphi_1\,dr = \frac{1}{k}\sum\alpha_i J_{i1}+\frac{1}{2}\alpha_2. \tag{25}$$

This equation coincides with (16) and thus the integral identity is automatically satisfied in the calculation of the phase shift by the Kohn method.

Along with this, if on the left side of Eq. (25) we substitute the coefficient α_2 for $\tan\eta$, we obtain the first equation of the set (10). If then follows that if the integral identity is satisfied in the phase calculation by Hulthén's method, then the set of equations (10) is consistent, the determinant (11) vanishes, and both the Hulthén method and the Kohn method give the same results. Hence a test of the fulfillment of the integral identity is equivalent to a direct comparison of the results of the calculation of the phase shift by the methods of Hulthén and Kohn, and therefore is not an independent criterion that confirms the validity of the variational calculation.

As an example, we have carried out a calculation of the phase η for the case of electron scattering by a static field of a hydrogen atom with

$$V = -2(1+1/r)e^{-2r}$$

in atomic units. In this case the following functions were used:

$$\varphi_1 = \sin kr, \quad \varphi_2 = (1-e^{-r})\cos kr, \quad \varphi_3 = e^{-r}(1-e^{-r})\cos kr; \tag{a}$$

$$\varphi_1 = \sin kr, \quad \varphi_2 = \cos kr - e^{-r}, \quad \varphi_3 = re^{-kr}. \tag{b}$$

The function (a) was employed in the work of Massey and Moiseiwitsch and was taken by us for comparison since it gives results close to the data of the numerical integration. The data for $\tan\eta$ computed by both methods, and also the values of $D = A_{12}^2 - A_{11}A_{22}$ in comparison with $k^2/4$ for each function are given in the table.

As is seen from the table, the values of D for the function (a) are close to $k^2/4$ and the values of $\tan\eta$ obtained by the two methods are practically identical. For the function (b), D differs considerably from $k^2/4$, and the value of $\tan\eta$ themselves, according to Hulthén and Kohn, differ significantly from each other and from the results obtained with functions (a), especially for k < 1. The function (b) evidently gives worse results for $\tan\eta$. However, at k = 1.7, the condition $D = k^2/4$ is approximately satisfied, and the values of $\tan\eta$, according to Kohn and Hulthén, coincide with the data of the numerical integration.

k	function (a)			function (b)			$\frac{k^2}{4}$
	$\tan\eta$		D	$\tan\eta$		D	
	according to Hulthén	according to Kohn		according to Hulthén	according to Kohn		
0.1	0.8783	0.8787	0,0026	0.4156	0.4184	0.0018	0.0025
0.15	1.2173	1.2174	0.0058	0.6138	0,6217	0.0036	0.0056
0,2	1.4248	1,4242	0.0104	0.7848	0.8009	0.0057	0.0100
0.3	1.7181	1,7189	0,0236	1.0373	1.0749	0,0107	0.0225
0.4	1,9076	1.9087	0.0429	1.1837	1,2399	0.0170	0.0400
0,5	1.7202	1.7211	0.0667	1.2564	1.3246	0.0252	0.0625
0,6	1,6285	1.6298	0,0964	1.2830	1.3546	0,0359	0.0900
0.8	1.4332	1.4353	0.1710	1.2630	1.3222	0.0692	0.1600
1,0	1.2722	1.2733	0.2616	1.1978	1.2343	0.1274	0.2500
1.2	1.1411	1.1404	0.3726	1.1192	1.1360	0.2244	0.3600
1.5	1.0031	0.9950	0.5533	0.9962	0.9978	0.4823	0.5625
2,0	0.8327	0.8356	0,9581	0.8266	0.8324	1.3807	1.0000

Thus Eq. (21), i.e., the condition of consistency of the initial system (10), can be regarded as a sufficiently reliable criterion for the validity of the choices of the variable function. If this condition is satisfied, then the integral identity (24) is also satisfied, and the results of the calculation of the phase by the methods of Hulthén and Kohn coincide.

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THE RELATION BETWEEN DECAY ASYMMETRY AND DIPOLE MOMENT OF ELEMENTARY PARTICLES

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Using a simple model for the emission of nonrelativistic particles, the wave function of a system is constructed in the coordinate representation for real and virtual decay. It is shown that asymmetry in real decay of a polarized particle depends on the imaginary part of the vector coupling constant, and that asymmetry in the virtual decay and the dipole moment both depend on the real part of this constant.

It is shown that elementary particles, which according to Landau have no dipole moment, resemble the enanthiomorphic (optically active) molecules of organic chemistry, rather than molecules in Λ-doubled states.

A consistent description of particle production is developed in the coordinate representation. A convenient expression is presented for the virtual decay probability.

THE assumption that parity is not conserved in weak interactions has led Lee and Yang[1] to new conclusions with respect to the behavior of elementary particles possessing spin. These conclusions are the following.

(1) Asymmetry decay is possible, in which the emitted particles are directed primarily along or against the angular momentum of the decaying particle.

(2) An elementary particle may have a dipole moment, and this is also parallel (or antiparallel) to its angular momentum.

As is well known, the first conclusion of Lee and Yang was brilliantly verified in experiments on the β-decay of oriented nuclei[2] and on μ mesons.[3] The order of magnitude of the dipole moment they predict, however, is too small for experimental observation. Landau[4] has given a complete theory relating parity nonconservation in charged-particle decay with space reflections.*

One of Landau's conclusions is that the dipole moment of elementary particles vanishes identically. It would seem at first that a decay asymmetry would lead necessarily to a dipole moment. Let us consider, for instance, a polarized neutron whose angular momentum is directed vertically upward. We may consider as established the fact that such a neutron decays by emitting electrons primarily in the upward direction. Let us now consider such a polarized neutron in the spherically symmetric field of a nucleus in which the energy relations are such that the neutron is stable and cannot decay. It then becomes possible and necessary for the neutron to undergo virtual decay, emitting an electron and capturing it again instantaneously. We may speak of a cloud of virtual electrons about the nucleus.

It would seem that asymmetry in real decay should correspond to a similar asymmetry in virtual decay. This would lead to asymmetry in the virtual electron cloud, and therefore to a dipole moment. Landau's work shows that such simple concepts are in general mistaken.† Ioffe[6] (who has kindly communicated his work to the present author before its publication) has recently shown that the decay asymmetry and dipole moment depend on whether or not the theory is invariant with respect to time reversal. Essentially the matter reduces to the following. A linear relation between the momentum p of the emitted particle and the direction of polarization (spin direction) s of the decaying particle is possible in a theory invariant under time reversal, since both quantities change sign. A static dipole d, however, or the analogous static center-of-mass r of a virtual-particle cloud, does not change sign under time reversal. Therefore the spin s can be related to the static quantities d and r only in a theory which is not invariant under time reversal.

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*Lee and Yang[5] have also independently indicated the possibility of combining space reflections with the transitions to the antiparticle.

†Note added in proof (November 25, 1957). In the absence of a dipole moment, parity nonconservation leads to certain specific magnetic properties, as is shown by the author on p. 1531 of the present journal (p. 1184 of translation).

Section 1 of the present article contains a coordinate-representation treatment of the emission of a nonrelativistic particle in the decay of a spin-$\frac{1}{2}$ particle, assuming parity nonconservation. This example shows clearly the dependence of the decay asymmetry of a polarized spin-$\frac{1}{2}$ particle on the phase of the coupling constants in the expression for the interaction leading to the decay.

It is shown that in the first approximation the decay asymmetry depends on the imaginary part of the vector coupling constant, and that the dipole moment depends on its real part, so that there is no direct connection between the decay asymmetry and dipole moment. It is shown how the interaction potential can lead to asymmetric decay in the case of a real coupling constant, i.e., in the Ioffe-Rudik-Okun' theory[7] (see assumption II of Ioffe[6]).

In Landau's theory charged particles are "odd." In molecular physics two types of odd phenomena are known. These are enanthiomorphic molecules (such as right and left tartaric acid) and diatomic molecules with Λ-doubling (such as a nitric oxide molecule with the projection of the election angular momentum on the axis directed from the N to the O, or the same molecule with the angular momentum directed in the opposite way). Section 2 explains the properties of such molecules with respect to their decay asymmetry and dipole moment. It is shown that according to Landau's theory elementary particles are similar to enanthiomorphic molecules, rather than to molecules with Λ-doubling.

In Sec. 1 it is found convenient to consider decay by treating the wave function of the produced particle in configuration space, rather than in momentum space. In Sec. 3 we show why it seems to us that this approach leads more simply and directly to the known formulas for the decay probability, without calculating the level density in phase space.

1. THE ASYMMETRY OF REAL AND VIRTUAL DECAY IN THE NONRELATIVISTIC MODEL

Let us consider the decay A = B + C, where A and C have spin $\frac{1}{2}$, and B has spin 0. We shall denote the states of A and C with $s_z = \frac{1}{2}$ by α, and those with $s_z = -\frac{1}{2}$ by β.† Let us assume that A and C are coupled by some spherically-symmetric potential field. We denote the wave function of A by χ, and that of C by η. We assume further that A and C are in a state with orbital angular momentum $\ell = 0$. We denote the wave function of B by φ. Let m be the mass of B, and assume that the decay energy is small compared with mc^2.

Assume that in the initial state A is polarized with $s_z = \frac{1}{2}$, so that the state is $\alpha\chi$. The wave function of particles B and C in the final state consists of two terms, and can be written

$$\Phi = \varphi_1\alpha\eta + \varphi_2\beta\eta \tag{1}$$

each term of which corresponds to one of the two possible polarizations of C after decay.

In nonrelativistic quantum mechanics there are two possible types of elementary interactions that do not involve the momenta of A and C. These are the scalar and vector (derivative) interactions. In the theory of second quantization, the corresponding terms and the Hamiltonian density H are

$$H' = g\psi_A^*\psi_C\psi_B + \text{hermitian conjugate}, \tag{2a}$$

$$H'' = f\psi_A^*\sigma\psi_C \operatorname{grad}\psi_B + \text{hermitian conjugate}, \tag{2b}$$

where g and f are coupling constants, ψ_A^*, ψ_B^*, and ψ_C^* are creation operators for particles A, B, and C, and σ is the spin operator.

Previously, the requirement that H be invariant under space conversion has led to the conclusion that it cannot contain a sum of these expressions. We now know, however, that in weak interactions, in which parity is not conserved, it is exactly such a sum H′ + H″ which must be considered.

The corresponding equations for the wave functions of B are the inhomogeneous Schrödinger equations (we set $\hbar = 1$)

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†In the case of real decay we can think of A as being a Λ particle, C a proton, and B a π meson. For virtual decay, on the other hand, we may consider the original particle A to be a proton, C a Λ particle, and B a meson, so that real decay is forbidden by energy considerations.

$$i\partial\varphi_1/\partial t = -(1/2m)\,\Delta\varphi_1 + V\varphi_1 + gq + f\partial q/\partial z, \tag{3}$$

$$i\partial\varphi_2/\partial t = -(1/2m)\,\Delta\varphi_2 + V\varphi_2 + f\,(\partial/\partial x + i\partial/\partial y)\,q, \tag{4}$$

where V is the potential of the forces acting on B, and

$$q = q\,(x,\ y,\ z,\ t) = \chi\eta^* = Q\,(r)\exp\{-i\,(E_A - E_C + M_Ac^2 - M_Cc^2 - mc^2)\,t\}. \tag{5}$$

Equations (3)–(5) are written specifically for the decay of a polarized particle A in which the wave functions χ and η are spherically symmetric, owing to the fact that $\ell = 0$. In what follows it is also important that Q(r) is real, which also follows from the fact that $\ell = 0$. The energies E_A and E_C do not include the rest masses.

We seek a stationary solution in which the time dependnence of φ_1 and φ_2 is the same as that of the inhomogeneous terms involving q in (3) and (4), namely

$$\varphi = \varphi\,(x,\ y,\ z)\,e^{-iEt},\ E = E_A - E_C + M_Ac^2 - M_Cc^2 - mc^2. \tag{6}$$

Inserting (6) into (3) and (4), we obtain

$$i\partial\varphi/\partial t = E\varphi = -(1/2m)\,\Delta\varphi + \ldots$$

Decay is possible when E > 0, and then B particles are emitted with momentum p, where $p^2/2m = E$. Far from the source, the solution should be a diverging wave $\varphi \sim e^{ipr}/r$.

If E < 0, real decay cannot occur. Virtual decay is described by a solution which decays exponentially with distance, such as $e^{-\kappa r}/r$, where $\kappa^2/2m = -E$.

Let us further simplify the problem, so that we may obtain an exact solution. We assume that the particle produced has no forces acting on it (V = 0) and that the region in which q is nonzero is small compared to the wavelength of B. Then for E > 0, we have

$$\varphi_1 = ae^{ipr}/r + b\cos\theta\frac{d}{dr}(e^{ipr}/r), \qquad \varphi_2 = be^{i\varphi}\sin\theta\frac{d}{dr}(e^{ipr}/r). \tag{7}$$

When pr ≫ 1,

$$\varphi_1 = (e^{ipr}/r)\,(a + ipb\cos\theta),\ \varphi_2 = (e^{ipr}/r)\,ipbe^{i\varphi}\sin\theta. \tag{7a}$$

For virtual decay, when E < 0, we have

$$\varphi_1 = a\,(e^{-\varkappa r}/r) + b\cos\theta\frac{d}{dr}(e^{-\varkappa r}/r) = (e^{-\varkappa r}/r)\left[a - \varkappa b\cos\theta\left(1 + \frac{1}{\varkappa r}\right)\right],$$
$$\varphi_2 = be^{i\varphi}\sin\theta\frac{d}{dr}(e^{-\varkappa r}/r) = -(e^{-\varkappa r}/r)\,e^{i\varphi}\varkappa b\sin\theta\left(1 + \frac{1}{\varkappa r}\right). \tag{8}$$

In Eqs. (7) and (8)

$$a = g\int Q\,(r)\,4\pi r^2dr,\ b = f\int\frac{dQ}{dr}4\pi r^2dr. \tag{9}$$

These expressions give information on the decay asymmetry and on the asymmetry of the virtual-particle cloud when decay does not take place. It is characteristic that there appear two terms in the expressions for φ_1: If particle A is in a state with $\ell = 0$ and $s_z = +\frac{1}{2}$, the final state in which C has $\ell = 0$ and $s_z = +\frac{1}{2}$ can have B both in the form of an S wave (with $\ell = 0$) and of a P wave (with $\ell = 1$ and $\ell_z = 0$). It is the interference of the two terms of φ_1 which leads to the terms linear in $\cos\theta$, which are those of interest in that they are related to the direction of the angular momentum of the decaying A particle. Here θ is the angle between the direction of polarization of A (the z axis) and the radius vector r. The wave function φ_2 corresponds to the decay of A with $s_z = \frac{1}{2}$ to C with $s_z = -\frac{1}{2}$ and describes particle B in the P state with $\ell = 1$ and $\ell_z = +1$.

Since φ_1 and φ_2 belong to different orthogonal states of C [see Eq. (1)], they do not interfere.

Let us find the flux of the B particles at large values of r. This is

$$\mathbf{j} = (1/2mi)\,(\overset{*}{\varphi}_1\nabla\varphi_1 - \nabla\overset{*}{\varphi}_1\cdot\varphi_1 + \overset{*}{\varphi}_2\nabla\varphi_2 - \nabla\overset{*}{\varphi}_2\cdot\varphi_2) = (p/r^2m)\,[|a^2| + p^2|b|^2 + \cos\theta\cdot ip\,(a^*b - ab^*)]. \tag{10}$$

For E < 0, we are interested only in the density ρ of virtual B particles, since at infinity the flux vanishes. This is

$$\rho = \overset{*}{\varphi}_1\varphi_1 + \overset{*}{\varphi}_2\varphi_2 = r^{-2}e^{-2\varkappa r}\left[|a|^2 + \varkappa^2|b|^2(1+1/\varkappa r)^2 - \varkappa\cos\theta\,(1+1/\varkappa r)\,(a^*b + ab^*)\right]. \qquad (11)$$

The fundamental result is contained in Eqs. (10) and (11), which show that there is no necessary relation between the decay asymmetry and the asymmetry of the virtual-particle cloud; one of these depends on $(a^*b - ab^*)$, and the other on $(a^*b + ab^*)$.

Let us assume the scalar coupling constant g to be real, so that a is real.

The decay asymmetry depends on the imaginary part of b, and vanishes if f is a real constant and b is real.

If, on the other hand, f is imaginary, and therefore so is b, the decay is asymmetric although the term containing $\cos\theta$ vanishes in the expression for the virtual particle density; the virtual-particle density is therefore spherically symmetric, and there is no dipole moment. Imaginary f corresponds to the Landau case, and real f to the Ioffe-Rudik-Okun' case (Ioffe's[6] assumptions I and II).

If for real f we consider the projection of the C spin along an axis perpendicular to the z axis (which is the polarization direction of the initial A particle), it is easily seen that from the interpretation of φ_1 and φ_2 one obtains the relation Ioffe predicts between the vector product $[S_A \times S_C]$ and the direction in which particle B is emitted.

We note that if we do not neglect the action of the potential on particle B, but assume it to be spherical, the conclusion that there is no dipole moment if f is imaginary remains valid. This follows from the fact that in the equation obtained from (3) the real term gQ gives a spherically symmetric real solution, the imaginary term $f\cos\theta \cdot Q'$ gives an imaginary solution proportional to $f\cos\theta$, and, as before, there are no interference terms proportional to $\cos\theta$ in the expression for the density.

Although for real f there is no decay asymmetry in the absence of a potential and when the source is small, a decay asymmetry does appear when V(r) is included. Then in the expression for the diverging wave, the S-wave and P-wave phases shift by different amounts α_S and α_P, and asymptotically when $pr \gg 1$ we obtain the expression†

$$\varphi_1 = a'\exp(ipr + i\alpha_S)/r + ip\cos\theta\cdot b'\exp(ipr + i\alpha_P)/r \qquad (12)$$

and real b gives

$$j = (p/mr^2)\left[a'^2 + p^2b'^2 + 2a'b'\cos\theta\sin(\alpha_S - \alpha_P)\right]. \qquad (13)$$

2. ODD MOLECULES

Let us consider the decay A = B + C from the point of view of many-body quantum mechanics. We shall consider A and C to be systems (molecules) in eigenstates with the same angular momentum L. The systems consist, for instance, of nuclei and electrons whose spins we do not take into account, considering only the Coulomb interaction. The theory is invariant with respect to reflection in space, as well as with respect to the transformation of time reversal accompanied by the transition to the complex conjugate wave function according to Wigner. If the states A and C are only (2L + 1)-fold degenerate in accordance with the possible L_z values for a given L, it is easily shown that neither for real nor for virtual decay will there appear terms linear in $\cos\theta$.

In the usual case (I) in which the internal states of A and C are uniquely determined, the eigenstates A_0 and C_0 with angular momentum L are just (2L + 1)-fold degenerate. In addition to this, we may consider enanthiomorphic molecules (II) which may be either right-handed (D) or left-handed (L). We shall designate these states by A_D and A_L respectively. Because spontaneous transitions between these states are possible, they are not, strictly speaking, eigenstates. During a time short with respect to the $A_D \rightleftarrows A_L$ transition time, however, A_D and A_L may be treated as separate eigenstates. It is important only that the A = B + C decay itself not lead to a transition between A_D and A_L. We assume that

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† In Eq. (12) a′ and b′ are proportional to g and f, respectively. The proportionality factors are real, but their expressions in terms of Q and dQ/dr are not as simple as (9). In the absence of a potential, but with an extended source Q(r) and real g and f, there is again no asymmetry.

The quantities α_S and α_P depend on the potential V(r). They represent the phase shifts between the S and P waves of the regular solution of the homogeneous equation, and the free-particle S and P waves

$$\psi_S = \sin pr/r, \quad \psi_P = \cos\theta\,(\cos pr/r - \sin pr/pr^2).$$

C occurs in two possible states, C_D and C_L, and that the only transitions possible are $A_D = C_D + B$ and $A_L = C_L + B$.

Finally, certain diatomic molecules (III) with fixed nuclei may, in the ground state, have a nonzero projection Λ of the electron angular momentum on the molecular axis. States which differ only in the sign of Λ have the same energy. We shall denote these states by n and m, writing A_n and A_m. We may consider the states A_n, A_m, C_n, and C_m approximately as eigenstates in the same sense as we did with A_D, A_L, C_D, and C_L. Case (II) may be called a static asymmetric molecule, and case (III) may be called a dynamic one. The symbol A now stands for the set consisting of A_0, A_D, A_L, A_n, and A_m.

Under a continuous rotation of coordinates in any of the three cases (I, II, or III), a given state (A_0, A_D, or A_n) with given values of L and L_z remains the same state (A_0, A_D, or A_n) with the same value of L, but with a different L_z. Under time reversal the wave functions of A_0 and A_D are transformed to their complex conjugates, but A_{+L_z} is transformed into A_{-L_z}, so that L remains invariant, but L_z changes sign.

Under a space reflection, A_0 remains invariant, being nondegenerate, A_D is transformed to A_L, A_n is transformed to A_m, etc., and L and L_z remain invariant.

As is seen from Sec. 1, the decay asymmetry and the dipole moment, which are of interest in the present case, depend on the interference of the S wave and P wave (with $\ell = 1$ and $\ell_z = 0$) of particle B. Two such waves can be obtained simultaneously in the decay $A(L, L_z) = C(L, L_z) + B$ with equal values of L and L_z for particles A and C, with $L \neq 0$.

Let us consider a Hamiltonian similar to (2a) and (2b), describing the decay A = B + C, and let us require that this Hamiltonian be invariant under coordinate rotations, space reflections, and time reversal. By treating continuous rotations under which states with different L_z transform into each other, one can show that the constant coupling A and C with the S wave of particle B cannot depend on L_z, and that the coupling constant for the P wave of particle B is proportional to L_z. The possible asymmetry between positive and negative directions along the axis is proportional to the polarization of A, or to the mean value of L_z.

By treating space reflection, we see that in case I either the S-wave amplitude or the P-wave amplitude can be nonzero, and that these waves cannot interfere. This proves the impossibility of decay asymmetry and the absence of a dipole moment in an ordinary nondegenerate moleucle in an eigenstate of the angular momentum.*

Time-reversal considerations can be used to show that in cases I and II the constant coupling A and C with the S wave of particle B must be real, and that for the P wave must be imaginary (we note that an imaginary f corresponds to a real coupling constant if one goes over from grad ψ_B to the momentum operator, writing $H'' = \sigma_{AC} p_B$).

It follows from this that enanthiomorphic (right-handed and left-handed) molecules also have no dipole moment, as is the case for ordinary molecules and atoms; polarized and enanthiomorphic molecules, however, may have a decay asymmetry.

In case III (Λ-doubling) time-reversal considerations lead to the conclusion that the P-wave coupling constant is real. As is well known, Λ-doubled molecules actually have dipole moments. It follows from Sec. 1 that in the first approximation there will be no decay asymmetry, although such an asymmetry will appear in the second approximation when one takes into account the interaction potential of B and C.

Parity nonconservation in elementary particle interactions can be thought of simply as an internal asymmetry of the elementary particles. Landau's theory, which forbids dipole moments, shows that this internal asymmetry is of the static kind, rather than analogous to Λ-doubling.

3. DESCRIPTION OF PARTICLE PRODUCTION IN CONFIGURATION SPACE

In Sec. 1 the inhomogeneous Schrödinger equation was used to treat particle production, which is usually treated by an expansion in eigenfunctions of the particle produced. In the absence of a potential, Eq.

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*By nondegenerate we mean that there is to be no degeneracy (as in case II and III) greater than the (2L + 1)-fold degeneracy in the angular momentum. According to the usual terminology, the dipole moment of a molecule such as HCl means the dipole moment with fixed nuclei, rather than in the free-rotation state treated by Yang and Lee, by Landau, and in the present work.

(7) is analogous to the Born approximation. In scattering theory we write $\psi = \psi_0 + \varphi$, where ψ_0 is the incident wave and φ is the scattered wave, and neglect $V\varphi$ (where V is the scattering potential), assuming it to be a second-order term. We then obtain for φ the inhomogeneous equation

$$i\partial\varphi/\partial t = -(1/2m)\,\Delta\varphi + V\psi_0.$$

We may attempt to find a solution to this equation by a plane-wave expansion of φ, or by using a Green's-function method, writing

$$\varphi(r_1) = \frac{m}{2\pi}\int V(r_2)\,\psi_0(r_2)\,(e^{ipr_{12}}/r_{12})\,d^3r_2.$$

The problem of particle production without taking $V(r)$ into account is entirely analogous to the first approximation of scattering theory.

Let us now use the inhomogeneous Schrödinger equation to treat the problem of particle production in a potential field. This problem has no analogy in scattering theory. The equation is of the form

$$i\partial\varphi/\partial t = -(1/2m)\,\Delta\varphi + V\varphi + q. \tag{14}$$

If q is the "source" in the equation for φ, then the equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}\varphi^*\varphi = \operatorname{div}\mathbf{j} + \frac{1}{i}(q\varphi^* - q^*\varphi) \tag{15}$$

shows that its yield is proportional to $q\varphi^* - q^*\varphi$, which means that it depends on the phase difference between φ and q. If Q and φ are both real,† no particles are produced.

For simplicity, let us consider particle production in the S state. Let $V(r)$ be spherically symmetric. The homogeneous equation has for $\ell = 0$, a regular solution ψ which is real, finite, at $r = 0$, and proportional to $\sin(pr + \alpha)/r$ for $pr \gg 1$. The phase shift α is determined by the regularity condition and by the potential. To be specific, let us normalize by writing

$$\psi_1 = \sin(pr + \alpha)/r \text{ for } pr \gg 1. \tag{16a}$$

The second linearly-independent solution ψ_2 of the second-order equation has a singularity at $r = 0$ (singular solution). We shall choose it to make the Wronskian equal to unity:

$$\psi_2 = \cos(pr + \alpha)/r \text{ for } pr \gg 1, \tag{16b}$$

$$r\psi_2\frac{d}{dr}(r\psi_1) - r\psi_1\frac{d}{dr}(r\psi_2) = 1 \text{ for all } pr. \tag{16c}$$

Let us now return to the inhomogeneous equation. Using $q = Q(r)\exp(-iEt)$ and $\varphi = \varphi(r)\exp(-iEt)$ we obtain from Eq. (3)

$$E\varphi + (1/2m)\,\Delta\varphi(r) - V(r)\,\varphi(r) = q(r). \tag{17}$$

As is known, the solution of (17) can be expressed by means of quadratures in terms of the solutions ψ_1 and ψ_2 of the homogeneous equation. This solution is found quite simply using the Lagrange method of Eq. (6), since the equations for the coefficients c_1 and c_2 are in this case independent.[8] We obtain

$$\varphi = c_1\psi_1 + c_2\psi_2 = \psi_1(r)\int^r Q(s)\,\psi_2(s)s^2ds + \psi_2(r)\int^r Q(s)\,\psi_1(s)s^2ds. \tag{17'}$$

The integrals of (17′) determine c_1 and c_2 up to a constant factor. We apply the condition $c_2(0) = 0$ to the coefficient of ψ_2 (which means that we require the solution to be regular). The coefficient ψ_1 is determined from the condition that when $pr \gg 1$, the solution must represent a diverging wave, so that $c_1(\infty) = ic_2(\infty)$. Thus the final solution in the form of (17′) with the required properties is

$$2m\varphi = -\psi_1(r)\int_r^\infty Q(s)\,\psi_2(s)s^2ds + \psi_2(r)\int_0^r Q(s)\,\psi_1(s)\,s^2ds + i\psi_1(r)\int_0^\infty Q(s)\,\psi_1(s)\,s^2ds. \tag{18}$$

For $pr \gg 1$, Eq. (18) gives

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†After eliminating the factor e^{-iEt}.

$$2m\varphi = \frac{1}{r} e^{i(pr+\alpha)} \int_0^\infty Q(s)\,\psi_1(s)\,s^2 ds. \tag{19}$$

From this we easily obtain an expression for the total particle flux (decay probability), namely

$$J = \frac{m}{\pi p} M^2, \quad M = 4\pi \int Q\psi_1 s^2 ds = \int Q\psi_1 dV. \tag{20}$$

This is the same as the usual expression

$$J = (2\pi/\hbar)\, M^2 dN/dE, \tag{21}$$

where M is the same matrix element as that given in (20), and dN/dE is the number of levels per unit energy interval.

Let us form the expression

$$J = -i\int (Q\varphi^* - Q^*\varphi)\,dV. \tag{22}$$

Inserting the expression for φ from (18) into this equation, we see that J involves only the imaginary part of φ, which is proportional to M.

The above derivation also shows why the decay probability involves the matrix element $\int Q\psi\,dV$ taken over the regular function ψ_1 which is finite for small r; this matrix element remains finite even if we let Q approach a δ-function.

On the other hand, for small r and when $Q = \delta$, the wave function φ of the produced particle will itself increase as r^{-1} if it is an S wave (and as $r^{-\ell-1}$ if $\ell > 0$), owing to the second term in (18). The question arises as to why the expression for the wave function does not enter into the matrix element. The answer is that particle production depends only on the imaginary part of φ (for a real source), and that only the real part increases as $1/r$; as is seen from (18), the imaginary part is regular.

The mathematical expectation value for the number of virtual particles is particularly simply obtained in the coordinate representation. This value may be thought of also as the fraction β of the time that particle A is in state B + C if there is not sufficient energy for real decay. For weak coupling this fraction is

$$\beta = \int \varphi^2 dV, \tag{23}$$

where φ is a solution such as (18) (decaying exponentially for large r) of an inhomogeneous equation such as (17) with $E < 0$. A particularly simple expression is obtained if $V(r) = 0$. In this case we have

$$\varphi(r_1) = -\frac{m}{2\pi}\int \frac{1}{r_{12}} e^{-\varkappa r_{12}} Q(r_2)\,d^3r_2, \tag{24}$$

so that (23) can be written

$$\beta = \frac{m^2}{2\pi\varkappa}\iint Q(r_1)\,Q(r_2)\,e^{-\varkappa r_{12}} d^3r_1 d^3r_2. \tag{25}$$

We note that (25) does not contain r_{12} in the denominator, so that the expression does not diverge as Q approaches a δ-function.

In the usual method of approach, nonrelativistic theory gives

$$\beta = (2\pi)^{-3}\int |C_p|^2 d^3p, \quad C_p = \frac{\int Qe^{ipr}d^3r}{(p^2+\varkappa^2)/2m}. \tag{26}$$

It can be shown that Eqs. (23) — (25) are identical with the more complicated expression (26).

In the decay of a free elementary particle A interacting with no other fields, the source Q becomes a δ-function, and the potential $V(r)$ describes the interaction of the decay products B and C with each other.

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SKIN EFFECT AND FERROMAGNETIC RESONANCE

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We consider the normal and the anomalous skin effect in ferromagnetic metals under conditions of ferromagnetic resonance. The constant magnetic field is taken to be perpendicular to the surface of the sample. The influence of exchange effects on the resonance line width and on the shift of the resonance frequency is taken into account.

I.

Ferromagnetic resonance absorption is observed when the frequency of the electromagnetic field incident on the surface of the ferromagnetic is nearly equal to the eigenfrequency of the precession of the magnetization vector **M** around the direction of the magnetic field **H**. This effect is described by the equation of motion of the magnetization vector **M** (Ref. 1) in a certain effective magnetic field,

$$\frac{1}{\gamma}\frac{\partial \mathbf{M}}{\partial t} = \mathbf{M}\times\left(\mathbf{H} + \frac{2A}{M_s^2}\nabla^2\mathbf{M} - \frac{\beta}{M_s}\,\mathbf{M}\times\mathbf{H}\right). \tag{1}$$

The first term on the right-hand side of (1) is the true magnetic field inside the specimen. The second term is the effective field due to exchange forces caused by the inhomogeneity of the magnetic moment **M**, and the third term is a relaxation term that describes the approach of the magnetization **M** to an equilibrium position along the field **H**; γ is the gyromagnetic ratio, and M_s the saturation value of the magnetization.

The relaxation term was introduced by Landau and Lifshitz[1] for a phenomenological description of damping processes. Up to the present the physical meaning of this term is not completely clear, since all existing theories lead to values of the dimensionless constant β that are rather low compared with experiment. In most cases, however, such a term gives a fairly good description of the experimentally observed effects that are connected with relaxation.

In ferromagnetic metals, however, there exists one mechanism which leads to a finite resonance line width. It is connected with the second term on the right hand side of Eq. (1) and plays an important role when this term is large, i.e., when the magnetization **M** is appreciably inhomogeneous in the skin layer. Ament and Rado[2] and Macdonald[3] have given the theory of this phenomenon for the case where there is a constant magnetic field parallel to the surface of the sample, and Rado and Weertman[4] observed experimentally an exchange broadening of the resonance line of this kind and determined in this way the value of the constant A for the substance investigated by them. In the present paper we consider the case where the constant magnetic field is perpendicular to the surface of the ferromagnetic metal.

It is clear that for pure metals effects connected with exchange broadening of the resonance lines will occur only at low temperatures. Lowering the temperature increases the mean free path ℓ_0 of the conduction electrons and with it the electrical conductivity σ; the inhomogeneity of the magnetic moment in the skin layer increases thus since the depth of the skin layer itself decreases. One can show that if the temperature of the ferromagnetic substance is sufficiently low the mean free path of the conduction electrons will become of the same order of magnitude as, or larger than the depth of the skin layer. In that case we cannot use the classical theory of the skin effect: the skin effect becomes anomalous.[5] We consider therefore in the present paper both the normal and the anomalous skin effect under conditions of ferromagnetic resonance.

An investigation of the skin effect in ferromagnetic metals makes it possible to determine for them the value of the constant A. Another way to determine this constant is to consider the magnetization of ferromagnetic substances at low temperatures.[6] From Bloch's theory it follows that

$$M_s = M_s^0 (1 - CT^{3/2}),$$

and knowing the constant C we can calculate A. Skin effect experiments, however, do not require such low temperatures as are necessary for Bloch's law to be valid. In addition, there are metals (for instance, cobalt) which show a phase transition and lose their ferromagnetism at temperatures where the Bloch law cannot be applied. Skin effect experiments allow us to determine the constant A also for high-temperature modifications of such ferromagnetic substances.

Experiments on the anomalous skin effect in ferromagnets enable us to determine for them the ratio ℓ_0/σ and to find out in this way to what extent we can apply in this case the normal representation of the electron theory of metals[7] which we are using in the present paper. One must note that the anomalous skin effect in ferromagnetic substances can easily be observed at higher temperatures than in other metals. This is connected with the fact that the expression for the skin depth δ contains the magnetic permeability, which is large, especially when the constant magnetic field H_z is much smaller than $4\pi M_s$, and when the frequency of the electromagnetic field is nearly equal to the resonance frequency.

We have also considered[8] the anomalous skin effect in a magnetic field which is parallel to the surface of the ferromagnet, but in that paper we did not take into account the role of exchange effects.

2. Our problem is described by Eq. (1), the Maxwell equations, and the transport equation for the conduction electrons in the metal:

$$\operatorname{curl} \mathbf{e} = -i(\omega/c)(\mathbf{h} + 4\pi\mathbf{m}), \qquad (2a)$$

$$\operatorname{curl} \mathbf{h} = (4\pi/c)\,\mathbf{j}, \qquad (2b)$$

$$\frac{1+i\omega\tau}{\tau} + v_z\frac{\partial f}{\partial z} + \frac{eH_0}{c}\left(v_x\frac{\partial f}{\partial p_y} - v_y\frac{\partial f}{\partial p_x}\right) = e\mathbf{ve}\frac{\partial F}{\partial \varepsilon}. \qquad (3)$$

In these equations the magnetic field **H** consists of a constant term $H_z\mathbf{i}_z$ ($\mathbf{i}_z$ is a unit vector along the z-axis which is directed into the metal, perpendicular to its surface), and of the field **h** of the electromagnetic wave, which is of frequency ω and which is propagated along the z-axis. The magnetization **M** is the sum of two terms of similar kind. We shall assume that $h/H_z \ll 1$ and that $m/M_s \ll 1$. We neglect the displacement current in Eq. (2b).

In Eq. (3), F is the equilibrium distribution function of the conduction electrons, f the small non-equilibrium part of the distribtuion function, **v** the electron velocity, and τ their relaxation time. We shall assume that the electron energy ϵ is an isotropic function of their quasi-momentum **p**. H_0 is the resultant constant magnetic field in which the conduction electrons of the ferromagnetic are placed. Wannier[9] has considered the problem of the magnitude of this field. The reflection coefficient of the electrons q at the surface of the metal gives us the boundary condition for Eq. (3); q = 1 corresponds to specular reflection and $q = 0$ to diffuse reflection; the last case is, apparently, actually realized.

We must solve Eq. (1) under the boundary conditions given in Ref. 2. In the case where the magnetic field is perpendicular to the metal surface, the boundary conditions are

$$dm_x / dz = 0, \quad dm_y / dz = 0 \quad \text{for } z = 0.$$

We replace further the components e_x and e_y of the electric field intensity by the quantities $e_\pm = e_x \pm ie_y$ and similar expressions for the components of the other vectors in (1) and (2). These equations will then be of the following form

$$\{\pm \omega / \gamma + H_z (1 \mp i\beta) - (2A / M_s)\, d^2 / dz^2\}\, m_\pm = M_s (1 \mp i\beta)\, h_\pm, \tag{4}$$

$$de_\pm / dz = \mp (\omega / c)\,(h_\pm + 4\pi m_\pm), \tag{5a}$$

$$dh_\pm / dz = \mp (4\pi i / c)\, j_\pm. \tag{5b}$$

In writing down (4) we have neglected products of the small quantities m and h.

Azbel' and Kaganov[10] obtained an expression for the electrical current density $j_\pm$ from the solution of the transport equation (3),

$$j_\pm (z) = \frac{2\pi e^2 p^2}{(2\pi\hbar)^3} \left\{ q \int_{-\infty}^{\infty} K\left(\frac{z - z'}{l_\pm}\right) e_\pm (z')\, dz' + (1 - q) \int_0^{\infty} K\left(\frac{z - z'}{l_\pm}\right) e_\pm (z')\, dz' \right\}, \tag{6}$$

where

$$l_\pm = l_0 / [1 + i\,(\omega \mp \Omega)\,\tau], \quad l_0 = v\tau, \quad \Omega = evH_0 / pc;$$

and where all quantities are evaluated on the Fermi surface. The kernel K is given by the equation

$$K\left(\frac{z}{l_\pm}\right) = \int_0^{\pi/2} \frac{\sin^3\theta}{\cos\theta} \exp\{- |z| / l_\pm \cos\theta\}\, d\theta. \tag{7}$$

It is now convenient for us to introduce the functions

$$n_\pm (z') = \int_{z'}^{\infty} m_\pm (z'')\, dz'',$$

which represent the components of the magnetic moment of that part of the sample which lies between the plane $z = z'$ and infinity. They are connected with the components of the electric field by the equation

$$[\pm \omega / \gamma + B_z (1 \mp i\beta) - (2A / M_s)\, d^2 / dz^2]\, n_\pm (z) = \mp (c/\omega)\, M_3 (1 \pm i\beta)\, e_\pm(z), \tag{8}$$

which can easily be obtained by substituting into (5a) the relation between $b_\pm = h_\pm + 4\pi m_\pm$ and $m_\pm$ which follows from (4),

$$[\pm \omega / \gamma + B_z (1 \mp i\beta) - (2A / M_s)\, d^2 / dz^2]\, m_\pm = M_s (1 \mp i\beta)\, b_\pm. \tag{9}$$

We now introduce the dimensionless parameter $\xi = z/\ell_0$ and the new functions $g_\pm(\xi) = n_\pm(\ell\xi)$. Substituting (8) and (4) into (5) and defining $g_\pm(\xi)$ as an even function for negative values of ξ $[g_\pm(-\xi) = g_\pm(\xi)]$, we obtain for the functions $g_\pm$ the following equations

$$\frac{d^4 g_\pm}{d\xi^4} - P_\pm \frac{d^2 g_\pm}{d\xi^2} = i\zeta \left[q \int_{-\infty}^{\infty} K\left(\frac{z - z'}{l_\pm}\right)\left(\frac{d^2 g_\pm}{d\xi'^2} - Q_\pm g_\pm\right) d\xi' + (1 - q) \int_0^{\infty} K\left(\frac{z - z'}{l_\pm}\right)\left(\frac{d^2 g_\pm}{d\xi'^2} - Q_\pm g_\pm\right) d\xi' \right]. \tag{10}$$

where

$$P_\pm = [H_z (1 \mp i\beta) \pm \omega / \gamma]\, M_s l_0^2 / 2A; \quad Q_\pm = [B_z (1 \mp i\beta) \pm \omega / \gamma]\, M_z l_0^2 / 2A; \quad \zeta = 3 l_0^2 / 2\delta_0^2, \quad \delta_0^2 = c^2 / 2\pi\omega\sigma,$$

and where σ is the electrical conductivity.

3. Solving Eq. (10) we can calculate the surface impedances,

$$Z_\pm = \pm (4\pi i / c) e_\pm(0) / h_\pm (0),$$

i.e., the quantities which are directly measured in ferromagnetic-resonance experiments. We shall assume that q = 1, since that case is mathematically the simplest one, while the expression for the impedances does not depend on q at all in the limit of the normal skin effect and depends only weakly on q in the case of the anomalous skin effect. We must solve Eq. (10) (for the case of q = 1) in such a way that $g_\pm(\xi) \to 0$ as $\xi \to \infty$. Following Reuter and Sondheimer[5] we introduce thereto the functions

$$\varphi_\pm(t) = \int_{-\infty}^{\infty} g_\pm(\xi)\, e^{-i\xi t}\, dt, \tag{11}$$

$$\varkappa_\pm(t) = \frac{1}{l_0}\int_{-\infty}^{\infty} K\left(\frac{z}{l_\pm}\right) e^{-izt/l_0}\, dz. \tag{12}$$

Performing respectively four and two integrations by parts, we get

$$\int_{-\infty}^{\infty} g_\pm^{\mathrm{IV}}(\xi)\, e^{-i\xi t} dt = t^4\varphi_\pm(t) + 2t^2 g_1^\pm - 2g_3^\pm, \quad \int_{-\infty}^{\infty} g''_\pm(\xi)\, e^{-i\xi t} dt = -t^2\varphi_\pm(t) - 2g_1^\pm, \tag{13}$$

where

$$g_3^\pm = g'''_\pm(0), \quad g_1^\pm = g'_\pm(0).$$

From Eqs. (10) to (13) we find the solution

$$g_\pm(\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}\, \frac{g_1^\pm[t^2 + P_\pm + i\zeta\varkappa_\pm(t)] - g_3}{t^4 + P_\pm t^2 + i\zeta\varkappa_\pm(t)(t^2 + Q_\pm)}. \tag{14}$$

If we assume that the first derivative $g_1^\pm$ is given, we can express $g_3^\pm$ in terms of it, using the boundary condition $n''_\pm(0) = 0$,

$$\eta_\pm = \frac{g_3^\pm}{g_1^\pm} = -i\zeta Q_\pm \int_0^\infty \frac{\varkappa_\pm(t)\, dt}{R_\pm(t)} \Big/ \int_0^\infty \frac{t^2 dt}{R_\pm(t)}, \tag{15}$$

where

$$R_\pm(t) = t^4 + P_\pm t^2 + i\zeta\varkappa_\pm(t)(t^2 + Q_\pm).$$

The values of $e_\pm(0)$ and $h_\pm(0)$ on the metal surface are determined by Eqs. (8) and (4), while the surface impedances are given by

$$Z_\pm = \frac{8i\omega l_0}{c^2} Q_\pm \left\{\frac{1}{P_\pm - \eta_\pm}\int_0^\infty \frac{t^2 + i\zeta\varkappa(t)}{R_\pm(t)} + \int_0^\infty \frac{dt}{R_\pm(t)}\right\}. \tag{16}$$

We see immediately that the impedance Z_- depends on the magnetic field H_z in resonance fashion. It increases considerably when $H_z \approx \omega/\gamma$ (resonance) and decreases strongly when $H_z \approx \omega/\gamma - 4\pi M_s$ (anti-resonance). At the same time the impedance Z_+ does not show resonance behavior and is a monotonic function of the field. We shall therefore be mainly interested henceforth in the behavior of the quantity Z_- as a function of H_z.

We must note that in most cases we can neglect in the right-hand side of Eq. (10) the second derivatives of $g_\pm$ compared with the terms $Q_\pm g_\pm$. Indeed,

$$d^2 g_\pm / d\xi^2 \sim g_\pm l_0^2 / \delta^2,$$

where δ is the skin depth. At the same time, of the two coefficients Q, the smallest value is reached by Q_- in the anti-resonance region, where it is of the order of $\beta \dot{B}_z M_s l_0^2/A$. Typical values of the quantities which enter in that expression are, for pure ferromagnetic metals, $\beta = 5 \times 10^{-2}$, $B_z \sim 10^4$ oersted, $M_s \sim 10^3$ oersted, $A \sim 10^{-6}$ erg-cm^{-1}, and $\delta \sim 10^{-5}$ cm. One can easily verify that even in the anti-resonance region the second quantity exceeds the first by one or two orders of magnitude. Their ratio becomes even larger away from anti-resonance. If we neglect the first term on the left-hand side of Eq. (10), the expression for the impedances simplifies slightly:

$$Z_{\pm} = \frac{8i\omega l_0}{c^2} Q_{\pm} \left\{ \frac{1}{P_{\pm} - \eta_{\pm}} \int_0^{\infty} \frac{t^2 dt}{r_{\pm}(t)} + \int_0^{\infty} \frac{dt}{r_{\pm}(t)} \right\}, \tag{16a}$$

where $r_{\pm}(t) = t^4 + P_{\pm}t^2 + i\xi\kappa_{\pm}(t)Q$, and

$$\eta_{\pm} = -i\zeta Q_{\pm} \int_0^{\infty} \frac{\varkappa_{\pm}(t)\,dt}{r_{\pm}(t)} \Bigg/ \int_0^{\infty} \frac{t^2 dt}{r_{\pm}(t)}. \tag{15a}$$

We shall make this simplification in the following.

4. Let the distance over which the electric field intensity varies significantly be much larger than the electron mean free path ℓ_0. We can then, following Ref. 5, substitute $\kappa_{\pm}(0) = \frac{4}{3}[1 + i(\omega \mp \Omega)\tau]$ for $\kappa_{\pm}(t)$ in evaluating the integrals which occur in (15) and (16). This is the limiting case of the normal skin effect, and the surface impedance is given by

$$Z_{-} = -\frac{4\pi\omega l_0}{c^2} \frac{Q_{-}}{P_{-} - t_1 t_2} \left(\frac{1}{t_1} + \frac{1}{t_2} \right), \tag{17}$$

where t_1 and t_2 are the roots of the equation

$$t^4 + P_{-}t^2 + 2il^2 Q_{-} / \delta_0^2 \,[1 + i(\omega + \Omega)\tau] = 0, \tag{18}$$

satisfying the condition $\operatorname{Im} t > 0$. The frequencies and magnetic field intensities used in ferromagnetic resonance experiments mean, in the region of the normal skin effect, that $(\omega + \Omega)\tau \ll 1$. If $(\omega + \Omega)\tau$ can be neglected compared to unity, Eq. (18) can be slightly simplified and becomes

$$t^4 + P_{-}t^2 + 2il_0^2 Q_{-}/\delta_0^2 = 0, \tag{19}$$

In the case where the exchange line broadening is not appreciable, it is possible to simplify Eq. (17) considerably. It is necessary for this to let the constant A tend formally to zero. One of the roots of (19) then tends to infinity as $A^{-1/2}$, but the other one remains finite. We get for the surface impedance Z_{-} the well known expression

$$Z_{-} = \frac{(4\pi i\omega\mu_{-})^{1/2}}{c\sigma^{1/2}} = \frac{(4\pi\omega|\mu_{-}|)^{1/2}}{c\sigma^{1/2}} e^{i\vartheta/2}, \tag{20}$$

where

$$\mu_{-} = P_{-} / Q_{-} = |\mu_{-}| e^{i(3\pi/2 + \vartheta)}.$$

Since $\operatorname{Im} \mu_{-} < 0$, $-\pi/2 < \vartheta < \pi/2$.

One sees easily that exchange effects play an important part only near resonance. Far from resonance, in particular near anti-resonance, they are unimportant and we can then use expression (20). To clarify Eq. (17) for the case where it includes these effects, we introduce, following Ament and Rado,[2] an equivalent magnetic permeability μ. We define μ as that quantity which, if substituted instead of μ_{-} into expression (17), leads to expression (20) without having to consider explicity exchange effects. If $H_Z \ll 4\pi M_S$ and the frequency ω is close to resonance, we have

$$\mu = \frac{4\pi M_s [H_z(1 + i\beta) - \omega/\gamma + 4(1 + i)(2\pi A / \delta_0^2)^{1/2}]}{[H_z(1 + i\beta) - \omega/\gamma + 2(1 + i)(2\pi A / \delta_0^2)^{1/2}]^2}. \tag{21}$$

From Eq. (21) it is clear that the exchange width of the resonance line ΔH is of the order $A^{1/2}/\delta_0$ and that exchange effects cause a shift of the resonance field of the same order of magnitude. Moreover, the ratio $\Delta H/H_Z$ increases when the field decreases so that the limiting case $H_Z \ll 4\pi M_S$, which we have considered in most detail, is the most interesting one from the point of view of studying exchange effects.

Finally we give in this section the expression for Z_{-} for the case where the line width is determined by the exchange effects, and the magnetic field is $H_Z = \omega/\gamma$.

$$Z_{-} = (32\pi^2 M_z^2 \omega / c^2 A \sigma^3)^{1/4} e^{i\pi/8}. \tag{22}$$

In this case $Z_{-} \sim \omega^{1/4}$.

5. We shall now consider the opposite limiting case of the ultra-anomalous skin effect. Let the distance over which the value of the electrical field strength $e(z)$ changes appreciably be much less than the electron mean free path ℓ_0. In the integrals occurring in Eqs. (15) and (16) we have then essentially $t \gg 1$. We can thus replace the functions $\kappa_{\pm}(t)$ in these integrals by asymptotic expressions valid for large values of t, viz.: $\kappa_{\pm}(t) \sim \pi/t$, as was also done by Reuter and Sondheimer.[5] We get then

$$Z_{\pm} = \frac{8i\omega l_0}{C^2} Q_{\pm} \left\{ \frac{1}{P_{\pm} - \eta_{\pm}} \int_0^{\infty} \frac{t^3 dt}{t^5 + P_{\pm} t^3 + i\pi\zeta Q_{\pm}} + \int_0^{\infty} \frac{t\,dt}{t^5 + P_{\pm} t^3 + i\pi\zeta Q_{\pm}} \right\}, \tag{23}$$

$$\eta_{\pm} = -i\pi\zeta Q_{\pm} \int_0^{\infty} \frac{dt}{t^5 + P_{\pm} t^3 + i\pi\zeta Q_{\pm}} \Bigg/ \int_0^{\infty} \frac{t^3 dt}{t^5 + P_{\pm} t^3 + i\pi\zeta Q_{\pm}}. \tag{24}$$

The surface impedances $Z_{\pm}$ do not depend on the electron mean free path ℓ_0 in the case of the ultra-anomalous skin effect. This can be verified by making in the integrals occuring in (23) and (24) the substitution $t = \ell_0 s$.

In the general case, however, Eqs. (23) and (24) are rather complicated, and we shall therefore again consider only two limiting cases. The first takes place when the exchange effects are not important, in other words, if the second term on the left-hand side of Eq. (10) is much larger than the first one. In order that this case be realized it is necessary to have $|P_-| \gg (\pi\zeta |Q|)^{2/5}$, or, stated in clear notation,

$$\left| H_z(1+i\beta) - \frac{\omega}{\gamma} \right| \gg \left[\frac{72\pi^4\omega^2 A^3\sigma^2}{c^4 M_s^3 l_0^2} \left| B_z(1+i\beta) - \frac{\omega}{\gamma} \right|^2 \right]^{1/5}. \tag{25}$$

This inequality is satisfied if either the field H_Z is sufficiently far from the value ω/γ, or the line width βH_Z is much larger than the exchange width. If inequality (25) is satisfied, the first term in Eq. (23) will be small compared to the second one and we get for the impedance the expression

$$Z_- = \frac{8i\omega l_0}{c^2} \int_0^{\infty} \frac{t\,dt}{t^3 + i\pi\zeta\mu_-} = \frac{16}{9} \left(\frac{\pi\sqrt{3}\,|\mu_-|^2\omega^2 l_0}{c^4\sigma} \right)^{1/3} e^{i2\vartheta/3}. \tag{26}$$

In the case where the opposite inequality $|P_-| \ll (\pi\xi |Q_-|)^{2/5}$ holds, we have

$$Z_- = \theta\,(M_z^4 l_0^3 \omega^2 / c^4 A^2 \sigma^3)^{1/5} e^{i\pi/5}, \tag{27}$$

where the dimensionless constant is given by

$$\theta = {}^8/_5\,(4\pi/27)^{1/5}(b_0 + b_1) = 5.51,$$

with

$$b_n = \left[\sin\frac{3\pi(4-n)}{5} + 3\sin\frac{\pi(4-n)}{5} \right] \Big/ \left(1 - \cos\frac{2\pi}{5}\right)\left(1 + \cos\frac{\pi}{5}\right).$$

The exchange width of the resonance line is of the order of magnitude $\Delta H \sim 10A^3\sigma^2\omega^2/c^4M_S\ell_0^2$. For sufficiently low temperatures ΔH is thus proportional to $\omega^{2/5}$ and does not depend on the temperature; this is a refinement of the statement by Kittel and Herring.[11] The shift of the resonance field is of the same order of magnitude. As in the case of the normal skin effect, the shift is in the direction of smaller fields.

6. The dependence of the impedance Z_+ on the magnetic field has a non-resonant character. The role of the terms responsible for the exchange effects in the expression for Z_+ is negligibly small since the inequality $|P_+| \gg (\pi\xi |Q_+|)^{2/5}$ is always satisfied. We have thus

$$Z_+ = \frac{(4\pi\mu_+\omega)^{1/2}}{c\sigma^{1/2}} e^{i\pi/4} \tag{28}$$

in the limiting case of the normal skin effect and

$$Z_+ = \frac{16}{9} \left(\frac{\pi\sqrt{3}\,\mu_+^2\omega^2 l_0}{c^4\sigma} \right)^{1/3} e^{i\pi/3} \tag{29}$$

in the case of the ultra-anomalous skin effect. These equations differ from the corresponding expressions of Ref. 10 for the impedance Z_+ only by the factor $\mu_+ = (\gamma B_z + \omega)/(\gamma H_z + \omega)$.

7. We shall now consider the assumptions on which the present paper is based. In the equation of motion for the magnetization vector (1) we did not include explicity the effective field of the magnetic anisotropy. This can be done in two cases. Firstly, in the case where the magnetic field H_z is much larger than the effective field, the latter can simply be neglected. Secondly, if, to a sufficiently good approximation, the magnetic field is directed along an axis of symmetry of the ferromagnetic cyrstal, one can consider the field H_z to consist of two terms, the ordinary magnetic field H_i and the effective field of the magnetic anisotropy H_a. Such axes of symmetry are, for instance, the hexagonal axis in a hexagonal crystal and the tetragonal or trigonal axes in a cubic crystal. We must note, by the way, that the magnetic field H_i in the ferromagnet is connected with the external magnetic field H_e through the relation $H_i = H_e - 4\pi M_s$.

As was pointed out by Karplus and Luttinger[13] the current density in a ferromagnetic metal is not completely determined by expression (6). In ferromagnetic substances there is also a stationary current which is proportional to the vector product $[\mathbf{M} \times \mathbf{e}]$, with a constant of proportionality which is practically independent of temperature. This current gives rise to the anomalous Hall effect in ferromagnetic materials.[14] However, at the frequencies and fields ordinarily used in ferromagnetic resonance experiments, and at low temperatures, this current plays a relatively small role. At high temperatures it can be taken into account by a formal change of the quantity Ω in (18). The product $\Omega\tau$ is also usually still small compared to unity.

If the value of the gyromagnetic ratio γ is known, it is convenient to use Eqs. (22) or (27) to evaluate the constant A. These are valid for the limiting cases of the normal and the ultra-anomalous skin effect, respectively, for a field $H_z = \omega/\gamma$. If the values of the constants γ and A and of the ratio ℓ_0/σ for the given substance are determined experimentally by investigating the behavior of the impedances in the corresponding limiting cases, we can evaluate all the parameters in expression (16). One can then calculate by numerical integration the impedances $Z_\pm$ in the intermediate case and thus compare in that case the theory with experiment.

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LETTERS TO THE EDITOR

THE ELECTRODISINTEGRATION OF NUCLEI AT HIGH ENERGIES

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IN this letter we calculate the cross section for the electrodisintegration of nuclei under conditions when it would be meaningless to expand the external field, which is given in Møller potentials, in terms of multipoles; this differs from the hypothesis in an earlier article of the author.[1] Such a case can occur when $|\mathbf{q}| = |\mathbf{q}_i - \mathbf{q}_f| > \hbar c/R$, where $\mathbf{q}_i/c$ and $\mathbf{q}_f/c$ are the momentum of the electron before and after scattering, and R is the nuclear radius.

The ground state of a nucleus (Z, A) will be described by an orbital wave function[2] and the final state by a plane wave, i.e., we shall use a Born approximation for the neutron ejected from the nucleus. The interaction of the nuclear system (core + neutron) will be described by a potential well with symmetrical symmetry. The radial wave functions are then expressible in terms of spherical Bessel functions and the matrix elements are easily calculated.

The calculation gives the following result for the cross section of the electrodisintegration of a nucleus[3] when the electron is scattered within the solid angle $d\Omega$ while transferring the energy $\Delta E = \epsilon_i - \epsilon_f$ to the nucleus, and the neutron escapes from the nucleus with momentum $\hbar\mathbf{k}_n = \mathbf{q}_n/c$ and energy $\Delta E - \epsilon_n$, where ϵ_n is the binding energy of the neutron in the nucleus, within the solid angle $d\Omega_n$:

$$\frac{d\sigma}{d\Omega d\Omega_n} = \frac{16}{2j+1}\left(\frac{e^2}{mc^2}\right)^2 \frac{|\eta|}{x_0^3(1+\beta_\alpha^2)}\left\{\frac{2j+1}{4\pi}\left[g_n^2 S_1 \Phi_l''^2 + \left(\frac{2Z}{A-1}\right)^2 S_2 \Phi_l^2\right] - \frac{2g_n Z}{A-1}\Phi_l \Phi_l' S_3^{(k)} a_l^{(k)}\right\} k_{n\alpha}^2 dk_{n\alpha},$$

$$\Phi_l(k_\alpha' x_0) = \frac{x_0^2}{\beta_\alpha^2 - k_\alpha'^2}\left[k_\alpha' j_l(\beta_\alpha x_0)\, j_{l-1}(k_\alpha' x_0) - \beta_\alpha j_{l-1}(\beta_\alpha x_0)\, j_l(k_\alpha' x_0)\right] + \frac{x_0^2}{1+k_\alpha'^2}\left[k_\alpha' k_l(x_0)\, j_{l-1}(k_\alpha' x_0) + k_{l-1}(x_0)\, j_l(k_\alpha' x_0)\right], \quad (1)$$

$$\Phi_l' = \Phi_l(k_\alpha'' x_0), \quad x_0 = \alpha R, \quad \beta = \hbar^{-1}\sqrt{2\mu(V_0 - \varepsilon_n)}, \quad \eta = -\left[j_{l+1}(\beta_\alpha x_0)\, j_{l-1}(\beta_\alpha x_0)\right]^{-1}, \quad g_n = -1{,}91.$$

Here the subscript α denotes division by $\alpha = \sqrt{2\mu\epsilon_n}/\hbar$ (for example, $\beta_\alpha = \beta/\alpha$); μ and m are the reduced mass of the nuclear system and the mass of the nucleon, respectively; V_0 and R are the depth and width of the potential well; $j_\ell(z)$ and $k_\ell(z)$ are spherical Bessel functions.[4] We have furthermore

$$S_1 = \frac{q_f}{q_i(q^2 - \Delta E^2)^2}\left[q^2(\varepsilon_i\varepsilon_f - \mu_e^2) - \frac{1}{4}(q^4 - \Delta E^4)\} + \Delta E^2 \varepsilon_i\varepsilon_f\right],$$

$$S_2 = \frac{\Delta E^{-2}}{2(q^2 - \Delta E^2)^2}\frac{q_f}{q_i}\left[2(\mathbf{q}'\mathbf{q})^2\varepsilon_i\varepsilon_f + \frac{1}{2}(Q_1^2 - (\mathbf{q}'\mathbf{q})^2)(q^2 - \Delta E^2) + 2(\mathbf{Q}_1\mathbf{q}_i)(\mathbf{Q}_1\mathbf{q}_f) - 2\mathbf{Q}_1(\mathbf{q}'\mathbf{q})(\varepsilon_f\mathbf{q}_i + \varepsilon_i\mathbf{q}_f)\right],$$

$$S_3^{(k)} = \frac{\Delta E^{-1}}{(q^2 - \Delta E^2)^2}\frac{q_f}{q_i}\left\{\left[(\varepsilon_i + \varepsilon_f)\mathbf{q}'\mathbf{q} + (-1)^k \mathbf{Q}_1(\mathbf{q}_i + \mathbf{q}_f)\right](\mathbf{q}_i \times \mathbf{q}_f)_k + \frac{(-1)^k}{2}(q^2 - \Delta E^2)(\mathbf{q}\times\mathbf{Q}_1)_k\right\},$$

where μ_e is the rest energy of the electron,

$$\mathbf{q}' = \hbar c\mathbf{k}' = \mathbf{q}_n - \mathbf{q}/A \quad \text{and} \quad \mathbf{Q}_1 = \Delta E\mathbf{q}' + (\mathbf{q}'\mathbf{q})\,\mathbf{q}/2mc^2.$$

The functions $a_\ell^{(k)}$, which depend on the angles, are given by

$$a_l^{(1)} = \pm\frac{1}{\pi}\sum_{m_j=-j}^{j}\left\{\frac{(l+1/2)^2 - m_j^2}{(2l+1)^2}\Theta_{lm_j-1/2}(\theta\varphi)\,\Theta_{lm_j+1/2}(\theta'\varphi')\sin\left[m_j(\varphi-\varphi') - \frac{1}{2}(\varphi+\varphi')\right]\right\},$$

$$a_l^{(2)} = \pm\frac{1}{\pi}\sum_{m_j=-j}^{j}\left\{\frac{(l+1/2)^2 - m_j^2}{(2l+1)^2}\Theta_{lm_j-1/2}(\theta\varphi)\,\Theta_{lm_j+1/2}(\theta'\varphi')\cos\left[m_j(\varphi-\varphi') - \frac{1}{2}(\varphi+\varphi')\right]\right\},$$

$$a_l^{(3)} = \frac{1}{\pi} \sum_{m_j=-j}^{j} \left\{ \frac{(l + {}^1/_2 \pm m_j)^2}{(2l+1)^2} \Theta_{lm_j - {}^1/_2}(\theta\varphi)\, \Theta_{lm_j - {}^1/_2}(\theta'\varphi') \sin\left(m_j - \frac{1}{2}\right)(\varphi - \varphi') \right\},$$

where the plus sign is taken for $j = \ell + \frac{1}{2}$ and the minus sign for $j = \ell - \frac{1}{2}$; $\Theta_{\ell m}(\theta\varphi)$ are normalized associated Legendre polynomials, and the angles θ, φ and θ', φ' fix the direction of the vectors $\mathbf{q}'$ and $\mathbf{q}'' = \mathbf{q}' + \mathbf{q} = \hbar c\mathbf{k}''$. When $q \ll q_n$, q' and q'' are almost independent of the angles and are given by

$$q' \approx q_n, \quad q'' \approx q_n, \quad q'q \approx -q^2/2. \tag{2}$$

In this case Φ_ℓ and Φ'_ℓ are also independent of the angles and the integration in Eq. (1) is greatly simplified. In this approximation the second term in the expression for Q_1 can also be neglected.

We note that when the neutron is ejected from an s state $a_\ell^{(k)} = 0$. Then in the energy region defined by (2) we obtain for the angular distribution of the neutrons the formula

$$d\sigma/d\Omega_n = A\cos^2\theta_n + B\cos\theta_n + C,$$

where θ_n is the angle between $\mathbf{q}_n$ and $\mathbf{q}_i$ and the coefficients A, B and C, after integration with respect to Ω and $k_{n\alpha}$, depend only on the initial energy ϵ_i of the electron.

In conclusion, the author considers it his pleasant duty to thank Professor V. S. Mamasakhlisov for his interest.

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LOCALIZATION OF NUCLEONS IN THE O_8^{16} NUCLEUS

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CERTAIN problems related to the structure of the nucleus call for a determination of the localization of the nucleons in the nucleus.[1,2] In the present work, an attempt is made to find the localization of the nucleons in the O_8^{16} nucleus by a method developed by Daudel.[3]

Let us consider a system of nucleons in a volume V in a state with a definite value of spin projection; p is the number of protons (neutrons) with spin $\frac{1}{2}$. We break up the space V into p volumes v_i such that one can determine in each of these the probability P_i of meeting a proton (neutron) with spin $\frac{1}{2}$. The volumes v_i are called "boxes" following Ref. 3. The same determination is also applied to spin $-\frac{1}{2}$. The quantity

$$\eta = \left(p - \sum_i P_i\right)/p$$

characterizes the absence of localization of nucleons for two chosen divisions of the space V into boxes. The smaller the value of η for a given volume, the better the division into boxes.

Let $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_A)$ be the wave function of the nucleus, then

$$P_i = C_p^A\, p \int_{v_i} d\mathbf{r}_1 \int_{V - v_i} d\mathbf{r}_2 \ldots \int_{V - v_i} d\mathbf{r}_p \int_V d\mathbf{r}_{p+1} \ldots \int_V d\mathbf{r}_A \psi^*\psi.$$

According to the oscillator model, the nucleus O^{16} is in an $(s_{1/2})^4\,(1p_{3/2})^8\,(1p_{1/2})^4$ state. The wave function of the nucleus is $\psi = \det\varphi_i \det\varphi_k$; φ_i is the wave function of a single proton in the oscillator model and the subscript k refers to a neutron. Let us break up the volume of the nucleus into four boxes corresponding to the four protons (neutrons) with spin $\frac{1}{2}$ $(-\frac{1}{2})$. From symmetry considerations we examine three divisions of the nuclear volume into boxes: (a) Three concentric spheres; (b) One sphere of radius a with three zones specified in the following manner:

$$0 \leqslant \theta \leqslant x, \quad x \leqslant \theta \leqslant \pi - x, \quad \pi - x \leqslant \theta \leqslant \pi$$

(where $a \leq r \leq \infty$, $0 \leq \varphi \leq 2\pi$); (c) The protons (neutrons) with spin $\frac{1}{2}$ $(-\frac{1}{2})$ are situated on the vertices of a tetrahedron. The second division corresponds to a geometrical localization of nucleons over s and p shells. The boundaries of the boxes are chosen to make ΣP_i a maximum for each division. For divisions (a), (b), and (c) the values of η are respectively 0.545, 0.724, and 0.756.

Thus the division of the volume of the nucleus into boxes by concentric spheres is the best of all those considered. The radii of the spheres are equal to 0.768 R, 1.023 R, and 1.316 R, where $R = 3.276 \times 10^{-13}$ cm. is the radius of the nucleus, determined from the maximum slope of the nucleon density distribution curve. For a radius of 0.768 R the nucleon density amounts to 88% of its maximum value, so that three boxes are located in the surface layer of the nucleus and one in the center. There is no geometrical localization of nucleons over s and p shells.

By determining the dimensions of a box we establish an upper limit for the diameter of the nucleon. The first spherical layer has the smallest transverse dimension; its thickness is 0.835×10^{-13} cm. Consequently, the radius of a nucleon cannot be greater than 4.18×10^{-14} cm. This is in good agreement with experiments on scattering of electrons by protons which indicate that the radius of a proton is $\sim 4 \times 10^{-14}$ cm (Ref. 4).

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TABULATED MASS DIFFERENCES

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RESULTS of high-precision mass spectrometer measurements of atomic masses of stable isotopes from iron to zinc are given in the work of Quisenberry, Scolman and Nier[1] with calculations of atomic masses of the radioactive isotopes. The atomic masses of the radioactive isotopes were calculated from the atomic masses of the stable isotopes with the use of β decay energies from King's tables[2] and reaction energies from the tables of Van Patter and Whaling.[3] A check of these calculations carried out by the author has led to the finding that King's tables are not sufficiently complete and has made possible calculation of the atomic masses of Mn^{55}, Mn^{56} and Fe^{55} with great accuracy and in better agreement with other experimental data. In King's tables, the value 3.65 ± 0.03 Mev given for the total energy of the $\mathrm{Mn}^{56} \rightarrow \mathrm{Fe}^{56}$ β transition is derived as a weighted mean from the data of three works.[5,6,8] The author knows of seven works in which are published measurements of the limit of the β spectrum and the energy of the γ quanta emitted on β decay of Mn^{56} (they are listed in the table). The weighted mean of all of these values yields 3.710 ± 0.011 Mev for the total energy of the β decay of Mn^{56}. Using this value, one can calculate the mass of Fe^{55} from the mass of Fe^{56} by way of $\mathrm{Fe}^{56} \rightarrow \mathrm{Mn}^{56} \rightarrow \mathrm{Mn}^{55} \rightarrow \mathrm{Fe}^{55}$. From the mass difference $\mathrm{Fe}^{55} - \mathrm{Fe}^{54}$, the binding energy of the neutron in Fe^{55} is calculated from mass spectrometric data; this turns out to be 9.296 ± 0.014 Mev. According to the measurements by Kinsey and Bartholomew[11] for the reaction $Fe^{54}(n, \gamma)\ Fe^{55}$, this neutron binding energy comes to 9.298 ± 0.007 Mev. The discrepancy comes to merely 12 ± 16 Kev. If one uses the energy of the β decay of Mn^{56} from King's table, the same discrepancy, as shown in Ref. 1 becomes much worse — 60 ± 30 Kev. Calculation of the atomic masses of isotopes of manganese and iron in different ways, with the use of improved values of the energy of the β decay of Mn^{56}, allows one to replace the masses given in the work of Quisenberry et al[1] by the following more reliable weighted mean values: $Mn^{55} = 54.955512 \pm 8$, $Mn^{56} = 55.956700 \pm 8$, $Fe^{55} = 54.955761 \pm 8$.

Correction to King's Table

Decay	Decay Data					Weighted mean value of the total energy of the decay	
	Form	Mev	Error	Method	Reference	Mev	Error
$Mn \xrightarrow{56} Fe$	β^-	2,88	1	M.s.	[4]	3.710	11
		2,86	5	M.s.	[5]		
		2,81	3*	M.s.	[6]		
		2.82	8	Scin.	[7]		
	γ	0,866	20*	M.s.	[8]		
		0,845	15	M.s.	[5]		
		0,822	8*	M.s.	[6]		
		0.845	10*	Scin.	[9]		
		0.845	10*	Scin.	[10]		

Remark. The errors are presented in units of the last significant figure of the energy. An asterisk next to the value for the error denotes that its value was determined by the compiler of the table, either because the experimenter did not quote it in the cited work, or because in the compiler's opinion the error is undervalued. The abbreviations for the measurement method are: M.s. — magnetic spectrometer, Scin. — scintillation recorder.

At the present time the experimental data on hand is very extensive. To calculate good values of atomic masses it is necessary to have as much of the data as possible, to compare and estimate their true accuracy, and then to select all the reliable values for the actual calculation. Incomplete use of all reliable experimental data, as seen from the example cited, often leads to questionable values of mass differences.

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CERTAIN SOLUTIONS OF THE EQUATIONS OF PLASMA HYDRODYNAMICS

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In the present communication we shall consider certain exact solutions of the hydrodynamic equations of cold plasma in the presence of an external magnetic field, and also in its absence. For the sake of simplicity we shall regard the ions as being at rest, but this restriction is not a fundamental one and may be easily removed.

In the case of one-dimensional flow in a plasma the solution of equations

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{e}{m}E; \quad \frac{\partial E}{\partial x} = -4\pi e(n - n_+); \quad \frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0 \tag{1}$$

(v is the speed, n the electron density, n_+ the ion density, E the electric field) may be sought in the form

$$v = e^{-t/t_0} v_0(z); \quad \frac{e}{m} x_0 E = e^{-t/t_0} \varepsilon_0(z); \quad \frac{n}{n_+} = n_0(z); \quad z = \frac{x}{x_0} e^{t/t_0}, \tag{2}$$

where x_0, t_0 are arbitrary constants. In order to elucidate the nature of this solution we make the substitution $y = \ln z$ and insert (2) into (1) after suppressing the index 0 of the functions v_0, ϵ_0, n_0:

$$V_0(v - v') + vv'e^{-y} = -\varepsilon; \quad \varepsilon' e^{-y} = (\omega_0 x_0)^2 (1 - n); \quad -V_0 n' + e^{-y}(nv)' = 0; \quad \omega_0^2 = 4\pi n_+ e^2/m; \quad V_0 = -x_0/t_0. \tag{3}$$

If y is now expanded into a series about some point x_1, t_1 ($x = x_1 + \xi$; $t = t_1 + \tau$) and the term $V_0 v$ is neglected the system (3) takes on the form:

$$(v - V)v' = -\varepsilon_1; \quad \varepsilon_1' = (\omega_0 x_1)^2 (1 - n); \quad -Vn' + (nv)' = 0: \quad \varepsilon_1 = x_1 \varepsilon / x_0; \quad V = -x_1/t_0. \tag{4}$$

It describes a wave propagating with a speed V; the form of the wave

$$c_1 + \sin^{-1}\frac{n_+/n - 1}{c_2} - \sqrt{c_2^2 - \left\{\left(\frac{n_+}{n}\right)^2 - 1\right]^2} = \omega_0\left(\frac{x}{V} - t\right) \tag{5}$$

agrees with that found by Akhiezer and Liubarskii[1] ($c_{1,2}$ are constants of integration).

In the presence of an external homogeneous and constant magnetic field H (directed along the z axis) the equations assume the form (all quantities are independent of φ, z):

$$\begin{aligned} &\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{e}{m}E - \frac{e}{mc}v_\varphi H; \quad \frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial nrv_r}{\partial r} = 0; \\ &\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi v_r}{r} = \frac{e}{mc}v_r H; \quad \frac{1}{r}\frac{\partial rE}{\partial r} = -4\pi e(n - n_+). \end{aligned} \tag{6}$$

In this case the following solution exists

$$v_r = \frac{r}{r_0}v_r^0(\xi); \quad v_\varphi = \frac{r}{r_0}v_\varphi^0(\xi); \quad E = \frac{r}{r_0}E^0(\xi); \quad n = n^0(\xi); \quad \xi = \frac{t}{t_0} + \ln\frac{r}{r_0}, \tag{7}$$

where the functions with the index 0 satisfy the system of ordinary differential equations obtained from (6) by means of the substitution (7).

A similar solution exists also in the case when only v_x, v_y, E_x differ from zero and all the quantities depend on x, t:

$$\begin{aligned} &\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} = -\frac{e}{m}E_x - \frac{e}{mc}v_y H; \quad \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} = \frac{e}{mc}v_x H; \\ &\frac{\partial n}{\partial t} + \frac{\partial nv_x}{\partial x} = 0; \quad \frac{\partial E_x}{\partial x} = -4\pi e(n - n_+). \end{aligned} \tag{8}$$

In this case

$$E_x = \frac{x}{x_0}E^0\left[\frac{t}{t_0} + \ln\frac{x}{x_0}\right], \quad n = n^0\left[\frac{t}{t_0} + \ln\frac{x}{x_0}\right]$$

($v_{x,y}$ vary in a manner similar to $E_{x,y}$).

The system (8) also has solutions in the form of waves in which all the quantities depend on the difference $x - Vt$; integration of the system of equations obtained from the system (8) in this case gives:

$$c_1 + \delta\sin^{-1}\frac{n_+/n - \delta}{c_2} - \sqrt{c_2^2 - \left[\left(\frac{n_+}{n}\right)^2 - \delta\right]^2} = \sqrt{\omega_0^2 + \Omega^2}\left(\frac{x}{V} - t\right), \quad \delta = (\omega_0^2 - \Omega^2)/(\omega_0^2 + \Omega^2); \quad \Omega = eH/mc. \tag{9}$$

Solution (9) [as well as (5)] becomes harmonic for $|1 - n/n_+| \ll 1$. One can therefore say that these solutions give the form of the wave of finite amplitude which arises when a wave is propagated in a plasma of slowly varying density n_+ in the direction of small n_+.

I thank M. A. Leontovich and R. Sagdeev for discussions.

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WAVE EQUATION FOR A FREE NUCLEON

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It is well known that so far no nucleon wave equation, even for the free nucleon, has been formulated. This is specifically due to the fact that the mass of the proton differs from that of the neutron. In this connection it is of interest to remark that the linearization of a second-degree wave equation leads in general to the equation

$$[\gamma^\nu \partial / \partial x_\nu - (m_0 c / \hbar) \exp (a\gamma^5)]\,\psi = 0. \tag{1}$$

Here m_0 and a are constants. Unfortunately, the mass operator $M = m_0 \exp(a\gamma^5)$ does not commute with the Hamiltonian that follows from (1). We therefore generalize (1) in the following manner:*

$$[\Gamma^\nu \partial / \partial x_\nu - (m_0 c / \hbar)\, I \exp (2aT_3)]\,\psi = 0. \tag{2}$$

The commutation relations for the operators Γ^ν, I, and T_3 are determined by the following postulates: (a) the Hamiltonian, the spin-projection operator, and the third component of the isotopic spin, T_3, form a system of commuting operators; (b) the relation between momentum and energy has the usual form.

This gives

$$[\Gamma^\nu, \Gamma^\sigma]_+ = 2\delta_{\nu\sigma}, \quad [\Gamma^\nu, T_3]_+ = 0, \quad [T_3, I]_+ = 0, \quad [\Gamma^\nu, I]_- = 0, \quad T_3 T_3 = 1, \quad II = 1. \tag{3}$$

As is well known, the components of the isotopic spin have to satisfy the commutation relations

$$[T_2, T_3]_- = iT_1, \quad [T_3, T_1]_- = iT_2, \quad [T_1, T_2]_- = iT_3. \tag{4}$$

The operators Γ^ν, T_k, and I have irreducible eight-by-eight representations:

$$\Gamma^\nu = 1^{\mathrm{II}} \times \gamma^\nu, \quad T_k = \sigma_k \times \gamma^5, \quad I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times 1^{\mathrm{IV}}. \tag{5}$$

Here σ_k are two-by-two spin matrices; 1^{II} and 1^{IV} are two-by-two and four-by-four unit matrices respectively. The sign × denotes the direct product of the matrices.

It follows from (2) that in the nonrelativistic limit the components of the wave function are characterized by the following quantum numbers.

Component of the wave function	Sign of the spin projection	Sign of the energy	t_3	Particle rest mass
ψ_1	+	+	$-1/2$	m_0e^{-a}
ψ_2	—	+	$-1/2$	m_0e^{-a}
ψ_3	+	+	$1/2$	m_0e^{a}
ψ_4	—	+	$1/2$	m_0e^{a}
ψ_5	+	—	$1/2$	m_0e^{a}
ψ_6	—	—	$1/2$	m_0e^{a}
ψ_7	+	—	$-1/2$	m_0e^{-a}
ψ_8	—	—	$-1/2$	m_0e^{-a}

One sees from the table that the eigenvalue $t_3 = -\frac{1}{2}$ corresponds to a particle mass $m_1 = m_0e^{-a}$ and $t_3 = \frac{1}{2}$ corresponds to a mass $m_2 = m_0e^{a}$. Taking $m_0 = 1837.38\ m_e$ (the average of the proton and neutron masses) and $a = 3\alpha/10\pi$, then m_1 and m_2 are equal to the proton and neutron mass respectively within the experimental errors for these masses,[1] and (2) can be considered as the wave equation of the nucleon.

It is easy to show that one can obtain from (2) two current vectors:

$$s_\nu^{(1)} = ic\,\varkappa\psi^* \Gamma^4 \Gamma^\nu \psi, \tag{6}$$

$$s_\nu^{(2)} = -ic\,\varepsilon\psi^* (T_3 - 1/2)\, \Gamma^4 \Gamma^\nu \psi. \tag{7}$$

The first can be interpreted as the specific nucleonic current (κ is the nucleonic charge) and the second as the electric current. From (7) we obtain the nucleon electric-charge operator

$$Q = -\varepsilon (T_3 - {}^1/_2), \tag{8}$$

which commutes with the Hamiltonian and with the spin operator. The eigenvalues of Q are equal to zero for the neutron states and ϵ for the proton states.

It should be mentioned that (2) is not invariant under improper Lorentz transformations. It follows from the pseudoscalar character of γ^5 that a reflection of the space coordinates changes a proton into a neutron and vice versa. This means that (2) is invariant under simultaneous reflection of space and isospin coordinates.

Equation (2) can also be used to describe the doublet of the Ξ hyperons. One then has to take $m_0 = 2588\ m_e$ while a has the same value as for the nucleons. This yields a mass of 2586 m_e for the Ξ^- and 2590 m_e for the Ξ^0.

It is interesting to note that one can also find for the operators Γ^ν, T_k, and I a twelve by twelve irreducible representation which allows a description of the triplet of Σ hyperons.

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*A more detailed derivation of (2) and a generalization for the case of weak interactions will be published in the Transactions (Trudy) of the Institute for Physics and Astronomy of the Academy of Sciences, Estonian S.S.R.

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CAPTURE OF ELECTRONS IN BETATRONS

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THE process of electron capture into betatron orbits has been studied by many authors, with various accelerators, and sufficient experimental material has consequently been compiled. A number of theoretical papers have been devoted to the subject of electron capture.[1-5] Although they were not fully successful in giving a complete explanation of the capture process, they still allow an estimate of the contributions of the different capture mechanisms and a comparison with experiment. However, no attention has been paid so far to the following process, which can contribute to the capture of electrons into betatron orbits. Consider the motion the electrons in a coordinate system moving along the equilibrium orbit with a velocity equal to the azimuthal injection velocity of the electrons. In this coordinate system the electrons will move towards each other. Since the electron velocities corresponding to radial motion are small, the collision probability will be sufficiently large. As a result of multiple scattering, electrons starting out with equal oscillation amplitudes will later acquire a gaussian amplitude distribution. Thus, the conditions for strong damping of the oscillations will statistically be fulfilled for a certain fraction of the injected electrons.

A rough estimate of this effect can be made in the following manner. If an electron makes an elastic collision at the time when its velocity is at maximum, then the amplitude of oscillation will decrease as

$$\rho_0' = \rho_0 \cos \psi, \tag{1}$$

where ψ is the scattering angle. The probability of single scattering at an angle ψ in traversing a distance dx through an electron gas of density N_0 is given by

$$P(\psi)\, d\psi = \pi \frac{e^4}{W_\rho^2} \frac{\cos(\psi/2)}{\sin^3(\psi/2)} d\psi N_0\, dx, \tag{2}$$

where W_ρ is the energy of the radial oscillations:

$$W_\rho = W_\theta (1 - n)(\rho_0 / R_0)^2. \tag{3}$$

The number of electrons missing the injector is given by

$$N_\gamma = NN_0\, dx\, 2\pi \frac{e^4}{W^2_{\rho\,\mathrm{av}}} \int\limits_{\psi_{\min}}^{\pi-\psi_{\min}} \frac{\cos(\psi/2)}{\sin^3(\psi/2)}\, d\psi = NN_0\, dx\, 2\pi \frac{e^4}{W^2_{\rho\,\mathrm{av}}} \left(\frac{1}{\sin^2(\psi_{\min}/2)} - \frac{1}{\cos^2(\psi_{\min}/2)} \right), \tag{4}$$

where N is the number of injected electrons.

We now consider a numerical example. We assume: injection energy $W_\theta = 10$ kev; injection current $i_{in} = 0.4$ amp, injection duration $\tau = 1.5\ \mu$sec, radius of the equilibrium orbit $R_0 = 20$ cm, instantaneous radius $R_t = R_0$, amplitude of radial oscillations $\rho_0 = 1$ cm, distance from the filament to the edge of the anode of the injector $\Delta = 0.1$ cm, field index $n = 0.65$, and area of the injected beam $S = 1\ \mathrm{cm}^2$. We then obtain number of injected electrons

$$N = i_{\mathrm{in}} \tau / e = 3.75 \cdot 10^{12};$$

velocity of the injected electrons

$$v_\theta = \sqrt{\frac{2W_\theta}{m}} = 5.92 \cdot 10^9\ \mathrm{cm/sec};$$

electron density in the beam

$$N_0 = N / v_\theta \tau S = 4.22 \cdot 10^8;$$

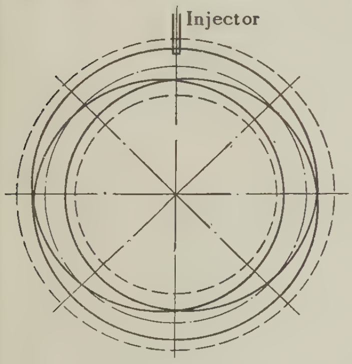
energy of the radial oscillations

$$W_\rho = 8.75\ \mathrm{ev};$$

minimum scattering angle necessary to miss the injector, $\cos\psi_{\min} = 0.9$.

For $R_t = R_0$ and $n = 0.65$, an electron leaving the injector travels radially a distance $8\rho_0$ before returning (see figure). Then the number of electrons missing the injector will equal $N_\gamma = 0.8 \times 10^9$, which corresponds, at the end of the acceleration, to an electron current in the betatron doughnut of

$$i_\gamma = (c/2\pi)\, e N_\gamma / R_0 = 0.03\ \mathrm{amp}.$$



We note that the number of electrons missing the injector increases as the square of the emission current of the injector. Taking into account the influence of the Coulomb repulsion, the number of captured electrons will be proportional to the fourth power of the emission current as the result of the decrease of the energy of the radial oscillations. This agrees with the experimental results.

In a more accurate treatment, one has to take it into account that the multiple scattering process is the predominant one. Furthermore, the amplitude of the radial oscillations can also be increased in collisions involving electrons of different velocities.

The authors are grateful to Professor M. S. Rabinovich for valuable suggestions and for discussion of the results.

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INTERPRETATION OF AN "ANOMALOUS" REPRESENTATION OF THE INVERSION GROUP

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THE purpose of this paper is to obtain a simple interpretation of an "anomalous" representation of the group of inversions of four-dimensional space-time. Gel'fand and Tsetlin[1] gave the name "anomalous" to a representation using operators which satisfy the condition

$$[T_{i'k'}T_{i''k''}]_+ = 0, \quad i, k = 0, 1, \quad i \neq k, \tag{1}$$

The inversion operators themselves of course commute.

Gel'fand and Tsetlin showed that, in the case of a scalar representation of the full Lorentz group, the anomalous representation is equivalent to the concept of parity doublets.[2] Our purpose here is to interpret the anomalous representation of the group of inversions of four-dimensional space-time in terms of a representation of the five-dimensional proper orthogonal group with pseudo-Euclidean metric.

The time-inversion T_{10} can be interpreted by means of a rotation operator H_{54} connecting the two half-spinors of four-dimensional space which represent a five-dimensional spinor. We may write

$$H_{54} = \exp\left(\frac{i\pi}{2} I_{54}\right), \tag{2}$$

where I_{54} is an infinitesimal operator of the representation O_5. It is easy to see that

$$T_{10} = H_{54} = \begin{pmatrix} 0' & i(-1)^{k+1} I \\ i(-1)^k I & 0' \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

We introduce a five-dimensional vector corresponding to the five-dimensional spinor. The vector is determined by a 4 × 4 matrix X which satisfies

$$X\psi = 0$$

and is defined by

$$X = \gamma_i x_i \ (i = 1, 2, \ldots, 5), \quad X^2 = x_5^2 + x_4^2 - x_1^2 - x_2^2 - x_3^2.$$

If

$$H_{54}^* X H_{54} = X', \tag{5}$$

then X' is defined by the coordinates $(-x_5, -x_4, x_1, x_2, x_3)$. Thus H_{54} corresponds correctly to the operation of time-inversion.

In the same way one can show that T_{11} defined by

$$T_{11} = H_{43} H_{21}, \quad H_{21} = \exp(i\pi I_{21}); \quad H_{43} = \exp(\pi I_{43})$$

corresponds to inversion of both space and time. Thus

$$T_{11} = \begin{pmatrix} i(-1)^k & 0' \\ 0' & i(-1)^{k+1} \end{pmatrix}. \tag{6}$$

Also

$$T_{01} = T_{11} T_{10}, \tag{6'}$$

defines T_{01} as the operator of space inversion $x_i' = -x_i \ (i = 1, 2, 3)$. There exists also an alternative anomalous representation in which T_{10} and T_{01} are respectively the space and time inversions. In this case Eq. (3), (6), and (6′) coincide with formulae given by Gel'fand and Tsetlin.[1]

The foregoing results are easily extended to any finite-dimensional representation of the full Lorentz group. One has only to use the fact that every representation of the rotation group is a direct product of an even or odd number of spinor representations.

We consider the connection of the anomalous representation with the four-dimensional Dirac equation.

We choose the anomalous representation in which T_{10} and T_{01} are respectively space and time inversions. Then T_{10}, T_{01}, T_{11} leave invariant only an extended 8-component Dirac equation

$$\Gamma_i \frac{\partial\psi}{\partial x_i} + m\psi = 0, \quad \Gamma_i = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}, \tag{7}$$

whereas the second variant of the anomalous representation leaves invariant a 4-component Dirac equation. The first variant of the anomalous representation also leaves invariant the Dirac equation for 4-spinors of the second kind[2]

$$\gamma_i \partial\psi / \partial x_i + m\gamma_5 \psi = 0. \tag{8}$$

Fermions with $m \neq 0$ which are described by the anomalous representation are in some ways reminiscent of the longitudinal neutrino which obeys the 2-component Dirac equation.[3] According to the choice of representation, they satisfy either a 4-component or an 8-component equation, and in the latter case T_{11} plays the role of γ_5 in the 2-component theory. It is possible that the eight components of the Dirac equation which describes "anomalous" Fermions may be connected with the existence of isotopic spin.

In conclusion we observe that Schwinger[4] proposed a connection between the Pauli exclusion principle and the lack of invariance of the Dirac equation under time-inversion. Ordinary time-inversion interchanges the two equations

$$\gamma_i \partial\psi / \partial x_i + m\psi = 0 \text{ and } \gamma_i \partial\psi / \partial x_i - m\psi = 0,$$

In the case of spinor particles described by the anomalous representation of the full Lorentz group (using the second variant, with T_{10} and T_{01} for time and space inversion respectively) the Lagrangian of the system is a true scalar under T_{10}. In other words, changing the order of factors in it does not change its sign. The requirement that the Lagrangian be invariant under the inversion T_{10} does therefore not lead to any additional restriction on ψ, and consequently does not imply the Pauli principle.

I thank Professors D. D. Ivanenko and M. L. Tsetlin for discussions.

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GALVANOMAGNETIC PROPERTIES OF MANGANESE FERRITE

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DATA on the galvanomagnetic properties of the ferromagnetic semiconductors — ferrites — are still quite scant. Yet a study of these properties is of interest from the point of view of clarifying the nature of the conductivity of materials of this type, and also from the point of view of establishing a connection between the electric properties and their antiferromagnetic state, since ferrites can be classified as "uncompensated" antiferromagnets.

We measured the temperature dependence of the galvanomagnetic effect in a ferrite consisting of 50% (mol.) MnO and 50% (mol.) Fe_2O_3. The resistance of this ferrite is not too high and the effect can therefore be measured with direct current over the range from room temperature to 350°C. The ferrite was prepared by the ordinary "ceramic" technology from chemically-pure oxides. The specimens were rods 52 mm long and 25 mm^2 in cross-section. Contacts were formed on the ends of the specimens for current carrying by burning-in a silver paste. The specimen was placed in a bifilarly-wound oven, placed in turn in a magnetized solenoid. The galvanomagnetic effect was measured with an unbalanced bridge. The same value dc current was passed through the specimen at all times. At each temperature, the specific magnetization σ was measured ballistically, in addition to measuring the electric resistivity and the galvanomagnetic effect $\Delta R/R$. All three quantities were determined practically simultaneously.

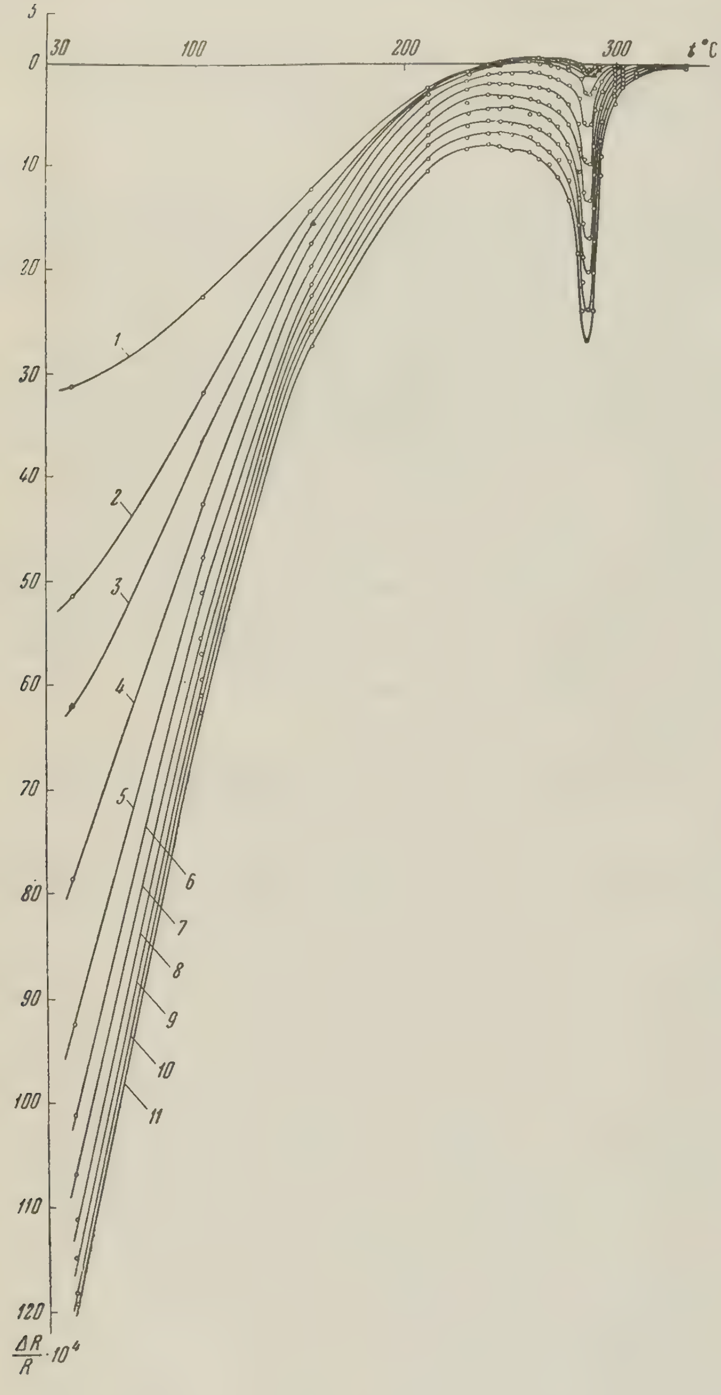

FIG. 1. Temperature dependence of longitudinal galvanomagnetic effect in manganese ferrite for different magnetic fields. 1 — 39.2; 2 — 65.4; 3 — 98.0; 4 — 196.0; 5 — 392.0; 6 — 654.0; 7 — 915.0; 8 — 1178.0; 9 — 1439.0; 10 — 1700.0; 11 — 1960.0 oersted.

Figure 1 shows curves of the temperature dependence of the longitudinal galvanomagnetic effect, plotted in weak and strong magnetic fields (above commercial saturation). It is seen that $(\Delta R/R)_{\parallel}$ has a negative sign and diminishes rapidly in magnitude, since the role of the displacement and rotation continuously diminishes with the temperature. Near the Curie point, however, the galvanomagnetic effect displays a sharp maximum, which owes its origin to the influence of the paraprocess. We note that Komar and Kliuskin[1] did not observe any influence of the paraprocess on the galvanomagnetic effect in nickel and nickel-zinc ferrites, probably because they did not make detailed measurements in the region of the Curie point. Our measurements have established that the dependence of the galvanomagnetic effect on σ and on H in the paraprocess region has approximately the same character as in metallic ferromagnets ($\Delta R/R$ linear with σ^2, $\Delta R/R$ linear with H in the region below the Curie point, $(\Delta R/R)_\Theta$ linear with $H^{2/3}$ directly at the Curie point itself, and $\Delta R/R$ linear with H^2 above the Curie point). It must be noted that the coefficient in the relation $(\Delta R/R)_\Theta = a_\Theta H^{2/3}$, which can serve as a quantitative characteristic of the influence of the paraprocess on the electric resistivity, has the same order of magnitude in manganese ferrite as in metals and alloys.[2] In manganese ferrite $a_\Theta \approx 20 \times 10^{-6}$ Oe$^{3/2}$, while in nickel $a_\Theta \approx 21 \times 10^{-6}$ Oe$^{3/2}$. In metallic ferromagnets, the value of a_Θ is determined by the conductivity of the tested ferromagnet. The greater the number of current carriers and the greater their mobility, the larger a_Θ (see Ref. 2). This explanation can apparently not be given for ferrites, since their conductivity is many times smaller than that of metallic ferromagnets, even at the Curie point.

An interesting feature of the galvanomagnetic effect in ferrites is that the longitudinal and transverse effects have the same sign, both being negative, while in metallic ferromagnets the longitudinal galvanomagnetic effect is positive in fields up to commercial saturation, while the transverse one is negative (exceptions to this rule exist).[3,4] In the case of manganese ferrite, however, we observed along with a negative galvanomagnetic effect also a positive component of the longitudinal effect, which appears in weak fields. Figure 1, and particularly Fig. 2, display clearly this positive component. It is seen from Fig. 2 that the greater portion of the galvanomagnetic effect is due to the paraprocess; the effect is negative and it is characterized by an exact fulfilment of the relation $\Delta R/R = c\sigma^2$. At lower magnetization, a "struggle" takes place between the negative galvanomagnetic effect (due to displacement, rotation, and the paraprocess) and the positive component of the longitudinal effect; this leads to a strong distortion of the galvanomagnetic-effect curve in this region. Cooling and repeated heating do not eliminate this anomaly. The presence of a positive component in the longitudinal galvanomagnetic effect is apparently due to our ferrite having a phase, in which it has a "quasi-metallic" behavior with respect to the galvanomagnetic effect. The presence of this phase is also seen on the temperature vs. electric resistivity curves. In accordance with previous experimental[1,5] and theoretical[6] data, the straight line log R(1/T) displays in our case a break in the region of the Curie point, corresponding to the fundamental "ferrite" phase. However, unlike the results reported in Refs. 1 and 5, manganese ferrite displays in the same region of temperatures an additional break, which apparently must be ascribed to the second phase.

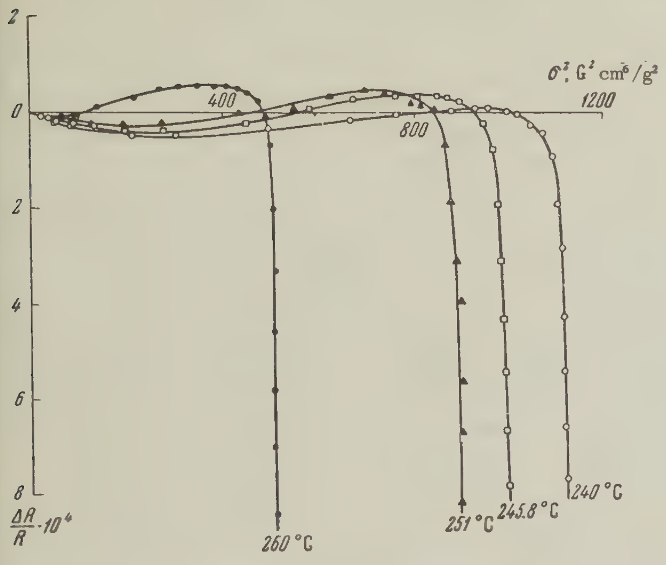

FIG. 2. Dependence of the longitudinal galvanomagnetic effect on the square of the magnetization in manganese ferrite.

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RELATIVE CROSS SECTIONS OF THE (n,p) AND (n,α) REACTIONS ON ELEMENTS WITH SEVERAL STABLE ISOTOPES

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THE available literature data on the cross sections of (n,p) and (n,α) reactions with 14 Mev neutrons are in general not in satisfactory agreement with the theoretical conceptions of these processes. Taking account of this circumstance, and owing to the importance of a correct interpretation of the mechanism of the (n,p) and (n,α) reactions for the explanation of the structure of atomic nuclei, an attempt was undertaken to establish an empirical law for the cross sections of these reactions as functions on the fundamental characteristics of the nucleus, by analysis of the data in the literature and by experiment.

It appears reasonable to expect that some sort of regularity must exist here and that its observation will contribute at least to the systematization of the knowledge of nuclear reactions, and possibly also to their explanation.

Even in a superficial examination of the data in the literature, it is striking that the cross sections for the (n,p) and (n,α) reactions in a series of stable isotopes of one element, decrease as a rule with increasing atomic weight of the isotope. In almost all cases this decrease is by approximately a factor of two. Such a primitive rule contradicts in general the modern ideas about the mechanism of nuclear reactions and, possibly precisely by virtue of its apparent theoretical inconsistency, it has not been taken into consideration.

With the aim of checking this rule and rendering it more precise, a series of experiments were set up to determine the relative cross sections of (n,p) and (n,α) reactions on elements with several stable isotopes, using 14-Mev neutrons. The cross sections were determined by the activation method.

In contrast to the majority of analogous works cited in the literature, the reaction products were analyzed here radiochemically, thus guaranteeing good reliability of the results obtained.

Preliminary results of the first experiments have been communicated earlier.[1] Data on new measurements and revised earlier data are presented below.

1. Relative cross sections of (n,p) reactions were measured for entire series of stable isotopes of nine elements. The following relative cross sections were obtained:

$$\sigma\,\mathrm{Ca}^{42} : \sigma\,\mathrm{Ca}^{44} = 1 : (0.24 \pm 0{,}02), \quad \sigma\,\mathrm{Ti}^{48} : \sigma\,\mathrm{Ti}^{49} = 1 : (0.55 \pm 0{,}15),$$
$$\sigma\,\mathrm{Zn}^{64} : \sigma\,\mathrm{Zn}^{66} : \sigma\,\mathrm{Zn}^{67} = 1 : (0.36 \pm 0.02) : (0.23 \pm 0.03),$$
$$\sigma\,\mathrm{Ga}^{69} : \sigma\,\mathrm{Ga}^{71} = 1 : (0.50 \pm 0.05),$$
$$\sigma\,\mathrm{Ge}^{70} : \sigma\,\mathrm{Ge}^{72} : \sigma\,\mathrm{Ge}^{73} : \sigma\,\mathrm{Ge}^{74} = 1 : (0.39 \pm 0.02) : (0.24 \pm 0.02) : (0.13 \pm 0.03),$$
$$\sigma\,\mathrm{Sr}^{86} : \sigma\,\mathrm{Sr}^{88} = 1 : (0.46 \pm 0.04),$$
$$\sigma\,\mathrm{Zr}^{90} : \sigma\,\mathrm{Zr}^{91} : \sigma\,\mathrm{Zr}^{92} : \sigma\,\mathrm{Zr}^{94} = 1 : (0.74 \pm 0.05) : (0.46 \pm 0.04) : (0.20 \pm 0.02),$$
$$\sigma\,\mathrm{Cd}^{106} : \sigma\,\mathrm{Cd}^{111} : \sigma\,\mathrm{Cd}^{112} : \sigma\,\mathrm{Cd}^{113} = (5 \pm 1) : 1 : (0.71 \pm 0.03) : (0{,}52 \pm 0{,}02),$$
$$\sigma\,\mathrm{Ce}^{140} : \sigma\,\mathrm{Ce}^{142} = 1 : (0{,}60 \pm 0{,}15).$$

The analogous ratios for four pairs of other isotopes, taken from data in the literature,[2,3] are presented below:

$$\sigma\,\mathrm{Mg}^{24} : \sigma\,\mathrm{Mg}^{25} = 1 : (0.23 \pm 0.10), \quad \sigma\,\mathrm{Si}^{28} : \sigma\,\mathrm{Si}^{29} = 1 : (0.46 \pm 0.17),$$
$$\sigma\,\mathrm{S}^{32} : \sigma\,\mathrm{S}^{34} = 1 : (0.23 \pm 0.10), \quad \sigma\,\mathrm{Fe}^{56} : \sigma\,\mathrm{Fe}^{57} = 1 : (\sim 0.5).$$

From the data presented it follows that the cross sections of (n,p) reactions on elements with several stable isotopes decrease sharply with increasing weight of the isotopes, i.e., with decreasing "relative concentration Z/A" of protons in the nucleus. This decrease is monotonic: in all cases the cross section is cut approximately in half upon reduction of Z/A by the equal amount $\Delta(Z/A) = 0.009$. Hence it is natural to suppose that this rule reflects a more general regularity, which relates the probability of emission of a proton from an excited nucleus, $\alpha_p = \Gamma_p/\Sigma\Gamma_i$, with the concentration of protons in the nucleus. An argument in favor of such a supposition is also the fact that similar relations are observed in many cases, at close excitation energies, for the cross sections of (γ, p) reactions[4] and for the quantities Γ_p/Γ_n, calculated by comparison of the cross sections of the (p, pn) and (p, 2n) reactions.[5]

The available experimental data on the absolute cross sections $(\sigma_{n,p})$ of (n,p) reactions and the cross sections (σ_c) for inelastic scattering of 14-Mev neutrons are very incomplete and insufficiently precise for a reliable check of the proposed hypothesis. The qualitative analysis of the data, however, favors this hypothesis, for with few exceptions, the values of $\alpha_p = \sigma_{n,p}/\sigma_c$, estimated from these data, decrease monotonically with decrease of Z/A [more precisely $Z/(A + 1)$] and the function $\alpha[Z/(A+1)]$ is close to a straight line on a semi-logarithmic scale.

2. Relative cross sections of (n, α) reactions were measured for four pairs of isotopes. The ratios obtained are presented below; the first ratio was calculated from the data of Paul and Clarke:[2]

$$\sigma Cl^{35} : \sigma Cl^{37} = 1 : (0.27 \pm 0.15),\quad \sigma Ge^{72} : \sigma Ge^{74} = 1 : (0.47 \pm 0.04),$$
$$\sigma Rb^{85} : \sigma Rb^{87} = 1 : (0.39 \pm 0.02),\quad \sigma Zr^{94} : \sigma Zr^{96} = 1 : (0.50 \pm 0.05),$$
$$\sigma Cd^{112} : \sigma Cd^{114} = 1 : (0.50 \pm 0.03).$$

As is seen from these data, the cross sections of the (n,α) reactions, like the cross sections of the (n,p) reactions, decrease with increasing weight of the isotope.

3. In the course of the work, several reactions not previously observed were realized, and some new information on radioactive isotopes — the products of these reactions — was obtained:

(a) As a result of the radiochemical study of the products of the reactions Ga (n,p) and Ge (n,α), the existence of the isomer Zn^{71*}, earlier obtained by an (n,γ) reaction from enriched Zn^{70} without chemical identification,[6] was corroborated. The half-life was traced several times in the course of 20 — 25 hours and was found to be 3.92 ± 0.05 hr instead of 3 hr as indicated in Ref. 6. It was found that the isomers Zn^{71*} (3.9 hr) and Zn^{71} (2.3 min) are formed through an (n,p) reaction on Ga^{71} in the ratio 2:3 and through an (n,α) reaction on Ge^{74} in the ratio 1:3. It is interesting that also another pair of isomers of zinc — Zn^{69*} (13.8 hr) and Zn^{69} (52.5 min), is formed with very close ratios (2:4 and 1:1) by the reactions Ga^{69} (n,p) and Ge^{72} (n,α).

(b) As a result of radiochemical analysis of the products of the Ge (n,p) reactions, the existence of the isotope Ga^{74} with $T = (8.0 \pm 0.5)$ min (Ref. 7) was confirmed.

(c) The half-life of Br^{84}, which is obtained by the reaction Rb^{87} (n,α), was found to be noticeably different from the value accepted in the literature on the basis of the analysis of bromine activities from fission products [(25.5 ± 0.5) min instead of 31 — 32 min].

(d) As a result of radiochemical analysis of the products of the Zr (n,α) reactions, the existence of Sr^{93} with $T = (7 \pm 0.5)$ min was confirmed. By means of absorption measurements, the maximum β-ray energy of Sr^{93} was found to be (3 ± 0.3) Mev.

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QUADRUPOLE MOMENTS OF THE EVEN ISOTOPES OF OSMIUM AND OF LUTECIUM 175

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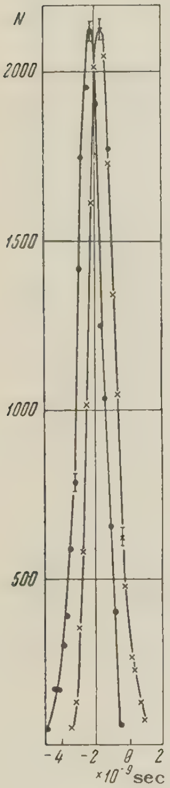
Two values are given in the literature for the half-life of the first excited state of the rotational band of the Os^{186} nucleus with an energy of 137 Kev: $T = 8.0 \times 10^{-10}$ sec (Ref. 1) and $T = 1.8 \times 10^{-9}$ sec (Ref. 2). In view of the fact that in this region of mass numbers the value of the quadrupole moment is observed to decrease with increasing number of neutrons in the shell, experiencing a jump on going from $N = 114$ to $N = 116$, it seems interesting to make the data on life-times more precise.



In the figure are presented coincidence curves, obtained with a Re^{186} source in the apparatus described in Refs. 3 — 5. The delay time is plotted along the abscissa and the number of coincidences along the ordinate. The points refer to measurements with the position of the β-ray absorber to the left of the source ($\gamma\beta$ coincidences) and the crosses to measurements with absorber to the right ($\beta\gamma$ coincidences).

From the displacement of the centers of gravity, the value $T = (6.0 \pm 0.2) \times 10^{-10}$ sec is obtained for the half-life; taking conversion into account, we obtain

$$T_\gamma = (1 + \alpha),\ T_{\text{exp}} = (1.44 \pm 0.05) \cdot 10^{-9} \text{ sec.}$$

for the radiation period.

Calculation according to Eqs. (29) and (31) of Bohr's article[6] gives $Q_0 = (6.4 \pm 0.1) \times 10^{-24}$ cm^2 for the nuclear quadrupole moment of Os^{186} (110 neutrons).

For Os^{188} (112 neutrons) $Q_0 = 5.1 \times 10^{-24}$ cm^2 (Ref. 7) and for Os^{190} (114 neutrons) $Q_0 = (4.2 \pm 0.9) \times 10^{-24}$ cm^2 (Ref. 8). Os^{192} (116 neutrons) is already a spherical nucleus.[9] These numbers illustrate the sharp transition from rotational structure to vibrational, already noted in the literature,[9] analogous to the long established jump-like change from vibrational structure to rotational on change of the number of neutrons from 88 to 90.[10]

At the same time, measurements were carried out on the lifetime of the first excited state of the rotational band of the Lu^{175} nucleus, for which an upper limit of $\tau \leq 2 \times 10^{-9}$ sec is given in Ref. 11. No clear relative displacement of the $\beta\gamma$ and $\gamma\beta$ coincidence curves appeared in our measurements. We estimate the upper limit of the lifetime of the 113-kev level to be 2×10^{-10} sec, or taking account of conversion, $\tau_\gamma \leq 6.8 \times 10^{-10}$ sec. Considering that the transition under discussion represents a mixture of transitions, 25% E2 and 75% M1, we obtain a partial lifetime τ_γ (E2) $\leq 1.4 \times 10^{-9}$ sec with respect to transitions of type E2.

Since the spins of the ground and first excited states of the Lu^{177} nucleus are 7/2 and 9/2 respectively, we obtain, by using Eqs. (29) and (33) of Ref. 6 for the case of E2 transitions with change of spin by unity, a lower limit for the value of the internal quadrupole moment, $Q_0 \geq 6.8 \times 10^{-24}$ cm^2. This result does not conflict with the value of the quadrupole moment for the neighboring nucleus Hf^{177}, which has an analogous rotational structure, for which according to our measurements[3] $Q_0 = 8.1 \times 10^{-24}$ cm^2.

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Translated by R. Eisner
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LIFETIME OF THE 264-Kev LEVEL OF Er^{167}

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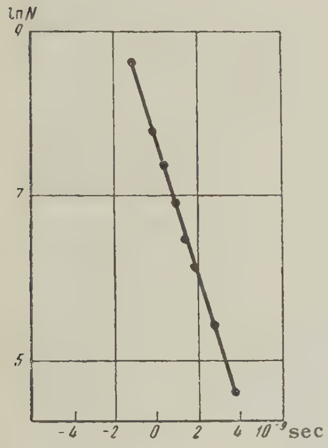
BY investigating coincidences between x-rays accompanying K-capture in Tu^{167} and internal conversion electrons for the 57-kev transition, by means of the apparatus described by Berlovich,[1-3] we have determined the lifetime of the 264-kev level of Er^{167}. The figure gives on a semilogarithmic scale the decay curve of the state under investigation (the delay time is plotted along the horizontal axis) from which the half-life is found to be

$$T_{1/2} = (2.0 \pm 0.5)\cdot 10^{-9} \text{ sec.}$$



Using the findings of Gromov and Dzhelepov,[4] that the 57-kev transition is a mixture of 25% E2 and 75% M1 transitions, and taking into account approximately the conversion in all the shells, starting with the L-shell (the conversion coefficient for the L-shell was computed with the aid of tables[5]), we have obtained for the half-life of radiative decay

$$T_\gamma = T_e(1 + \alpha) = 1.4\cdot 10^{-8} \text{ sec},$$

which yields for the partial decay periods the values:

$$T_\gamma(E2) = 5.6\cdot 10^{-8} \text{ sec and } T_\gamma(M1) = 1.87\cdot 10^{-8} \text{ sec.}$$

Comparison with the results calculated by means of Weisskopf's formula,[6] based on the concepts of the single particle model, leads to an acceleration factor F = 310 for the transition of E2 type, and to a retardation factor F = 1/160 for the transition of M1 type.

According to Gromov and Dzhelepov,[4] the first excited level (207-kev) is a $\frac{1}{2}-$ state, while the 264-kev level may have the characteristics $\frac{1}{2}+$, $3/2+$ or $3/2-$; according to Gorodinskii et al.,[7] the most likely characteristic is $3/2-$. The assumption that the strongly deformed Er^{167} nucleus (17 neutrons outside a filled shell) has a rotating band with an angular momentum component $\Omega = \frac{1}{2}$ along the axis of elongation leads to a value of the internal quadrupole moment which is approximately twice as big as the values observed in this region of mass numbers.[8] However, the possibility is not excluded that the discrepancy is connected with the inaccuracy in the relative amounts of E2 and M1 transitions proposed by Gromov and Dzhelepov,[4] and that in actual fact the proportion of transitions of type E2 is considerably smaller.

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[2] E. E. Berlovich, Izv. Akad. Nauk SSSR, Ser. Fiz. **19**, 343 (1955).

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Translated by G. Volkoff
310

CHANGE IN THE TEMPERATURE OF ANTIFERROMAGNETIC TRANSFORMATION OF MANGANESE TELLURIDE UNDER PRESSURE

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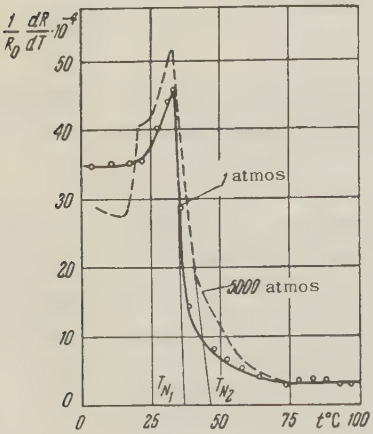
THE effect of the shift of the Curie point of ferromagnetic substances under the influence of hydrostatic compression has been investigated repeatedly. However, up to the present time there has not been a single paper devoted to the experimental investigation of the influence of hydrostatic pressure on the Néel temperature (T_N) of antiferromagnetic substances. By measuring the temperature and pressure coefficients of electrical resistance of manganese telluride we have investigated the influence of hydrostatic compression on the temperature of the antiferromagnetic transformation ($T_N = +37°C$)[1,2] of this compound.

Uniform hydrostatic pressure was applied to the sample in a high pressure chamber into the upper part of which four special electric leads were introduced. Transformer oil served as the medium for transmitting the pressure. In order to eliminate the effect of junction resistances at the boundaries of the sample and the metal electrodes supplying the current, a compensation method of measuring electrical resistance by means of probes was employed. Thin constanan wire served for the potential probes, while copper wire was used for the current leads. This enabled us to measure the temperature of the sample by means of a copper-constantan thermocouple without introducing additional electrodes. The cold junction of the thermocouple was at atmospheric pressure and at 0°C. Measurements were carried out over the temperature range of 279 — 363°K and the pressure range of 1 — 5200 kg/cm^2.



It has been established that hydrostatic compression results in a decrease of electrical resistance of manganese telluride. The value of the pressure coefficient $R_T^{-1}\,dR/dP$ varies as a function of temperature within the range -3.5 to -0.73. At temperatures far removed from the Néel temperature the electrical resistance varies linearly with the pressure. However, close to the temperature of the magnetic transformation the nature of the R(P) curves alters appreciably: a curvature becomes apparent with the R(P) curves being convex downward below T_N and upward above T_N. The figure shows the curve of the dependence on the temperature of the temperature coefficient of electrical resistance measured at atmospheric pressure and of the quantity $R_0^{-1}\,dR/dT$ calculated for a pressure of 5000 atmos. The calculations were carried out in a manner similar to that used for the determination of the shift of the Curie temperature of ferromagnetic substances.[3]

As can be seen from the figure, hydrostatic compression results in an increase of the temperature of antiferromagnetic transformation of manganese telluride. The magnitude of this effect is equal to $dT_N/dP = (2.0 \pm 0.4) \times 10^{-3}$ deg/kg/cm^2. This result was confirmed by direct measurements at a pressure of 4400 kg/cm^2. The Néel temperature determined from a break in the R(T) curve is in this case equal to +46°C.

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311

CONCERNING THE STABILITY OF SHOCK WAVES

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THE stability of normal discontinuities in an arbitrary medium with respect to perturbations of a wave type was studied by D'iakov.[1] He found a region of absolute instability when the perturbation from the shock grew with time, a region of stability when this perturbation is damped and, finally, a peculiar region of "spontaneous sound emission by the discontinuity," in which the perturbation has the form of an undamped travelling wave. However, the last region was not completely determined by D'iakov. In order to separate off the wave emitted by the discontinuity from the incident one, it was required in Ref. 1 that the wave vector $\mathbf{q}(q \sin \vartheta, q \cos \vartheta, 0)$ of the wave be directed away from the discontinuity.* Together with the requirement of the reality of $\mathbf{q}$ and ω this gives the condition

$$0 < \cos \theta < 1, \tag{1}$$

according to which the region of spontaneous emission was defined. But here the "transport" of the perturbation of the moving fluid was not taken into account.

We write the conditions of spontaneous emission

$$\operatorname{Im} \omega = 0, \quad \operatorname{Im} q_y = 0, \quad V_y > 0, \tag{2}$$

where $\mathbf{V}$ is the velocity of the perturbation in the system in which the discontinuity is at rest. Since, in this system, the fluid behind the shock moves with a velocity $\mathbf{v}(0, v, 0)$ and the velocity of the perturbation in the moving medium is[2]

$$\mathbf{V} = \mathbf{v} + \mathbf{q}\, c/q, \tag{3}$$

the Eqs. (2) lead to the inequality

$$-M < \cos \theta < 1, \tag{4}$$

where $M = v/c$ is the Mach number.

Thus, to region (1) (in which the emitted wave moves relative to the stationary fluid in a direction opposite to the moving shock wave) is joined the region

$$-M < \cos \theta < 0, \tag{5}$$

in which the emitted sound waves move in the same direction as the shock wave, gradually falling behind it.

Application of Sturm's theorem to the equation for $\cos \vartheta$ (see Ref. 1) gives us for the region (5)

$$\frac{1-M^2-v_0M^2/v}{1-M^2+v_0M^2/v} < \left(\frac{\partial V}{\partial p}\right)_H j^2 < \frac{1+2M^2-v_0M/v}{1+v_0M/v}. \tag{6}$$

The whole region of emission is completely determined by the inequalities

$$\frac{1-M^2-v_0M^2/v}{1-M^2+v_0M^2/v} < \left(\frac{\partial V}{\partial p}\right)_H j^2 < 1+2M. \tag{7}$$

It is of interest that the inequality (5) extends the region of emission in that direction which is most convenient for experiments, because it corresponds to the least exotic form of the Hugoniot adiabatic.

The condition that the left-hand side of the inequality (7) be negative and, consequently, that the region sought can, in principle, exist on ordinary adiabatics (where $(\partial V/\partial p)_H < 0$), is

$$1 < M^2(1+v_0/v). \tag{8}$$

This inequality is realized even for the case of an ideal gas. The region of sound emission is not however, present because the requirement that $(\partial V/\partial p)_H j^2 \equiv -M_0^{-2}$ (for an ideal gas) satisfy (7) leads to $M_0 < 1$, i.e., (7) coincides with the region of instability.

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*In the notation of Ref. 1 the components of the wave vector are $\mathbf{q}(k, \ell_2, 0)$. The remaining notation is unchanged here.

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REFLECTION AND REFRACTION OF SOUND BY A SHOCK WAVE

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FOR an ideal gas the normal incidence of sound on a shock wave has been treated by Blokhintsev[1] and by Burgers,[2] while Brillouin[3] has considered oblique incidence. However, in the last reference the author obtains erroneous results for the reflection and transmission coefficients as he does not take into account the perturbation of the surface of the shock wave.

The reflection and refraction of sound by a shock wave in an arbitrary medium is considered in this note.

As is well known, the shock wave moves with a supersonic speed with respect to the rarefied medium and with a subsonic speed with respect to the compressed medium. Therefore if a sound wave is incident on the surface of discontinuity from the side of the compressed medium, no transmitted wave is formed, and a reflected wave and an entropy-rotational perturbation carried by the liquid flux appear. Similarly, when sound is incident on the shock wave from the side of the rarefied medium no reflected wave is formed while an entropy-rotational wave appears in addition to the transmitted wave. A perturbation wave of amplitude η moves along the surface of discontinuity with a propagation vector $\mathbf{k}$ equal to the projection of the propagation vector of the incident wave $\mathbf{k}_i(k, \ell, 0)$ on the surface of discontinuity and with a frequency ω equal to the frequency of the incident sound wave in the system of coordinates in which the surface of discontinuity is at rest.

The equations for the entropy-rotational (1) and the sound (2) disturbances have the form:*

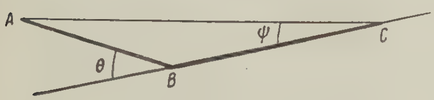
$$k\delta v_x^{(1)} + l^{(1)}\delta v_y^{(1)} = 0;\ \delta p^{(1)} = 0;\ l^{(1)}v = \omega; \tag{1}$$

$$\delta v_i^{(2)} = k_i/\rho\,(\omega - vl^{(2)});\ \delta s^{(2)} = 0;\ (\omega - vl^{(2)})^2 = c^2\,(k^2 + l^{(2)2}). \tag{2}$$

We must write down the boundary conditions in the system of coordinates in which the normal velocity of the perturbed discontinuity is zero. Linearized with respect to the amplitude of the sound wave they give:

$$[v\delta\rho + \rho\delta v_y + i\omega\rho\eta] = 0;\ [\delta v_x + ik\eta v] = 0;\ [\delta p + v^2\delta\rho + 2\rho v\delta v_y] = 0;\ [\delta w + v\delta v_y + i\omega v\eta] = 0. \tag{3}$$

Equations (1) – (3) form a complete system. In the problem of reflection we must distinguish between the incident $\delta A^{(i)}$ and the reflected $\delta A^{(r)}$ waves. At the same time from the side of the rarefied medium $\overline{\delta A} = 0$ which corresponds to unperturbed flow in front of the surface of discontinuity. In place of one of equations (3) it is more convenient to use a consequence of Hugoniot's equation[4]



$AB/BC = M$

$$\delta\rho = Q\delta p,\ Q = (\partial\rho/\partial p)_H. \tag{4}$$

It is convenient to express the result in terms of the angle $\theta = \tan^{-1}(k/l)$ and the auxiliary angle ψ whose definition is evident from the diagram. The law of refraction can be formulated as the equality $\psi^{(i)} = \psi^{(r)}$:

$$\delta p^{(r)}\{\cos(\theta^{(r)} - \psi) + h\} = -\delta p^{(i)}\{\cos(\theta^{(i)} - \psi) + h\},\quad h = \frac{\cos\psi}{2M}\left\{1 + Qv^2 + \frac{\bar{v}}{v}(Qv^2 - 1)\tan^2\psi\right\}. \tag{5}$$

In the refraction problem we are interested in the ratio of a quantity in the transmitted wave δA to the corresponding quantity in the incident wave $\overline{\delta A}$. The law of refraction is formulated as the "law of tangents"

$$\tan\psi/\tan\bar{\psi} = v/\bar{v},\quad \delta p\,\{(1 + M\cos\theta) - G\,(1 + 2M\cos\theta + M^2)\} = \delta\bar{p}\,\{\bar{M}\,(\bar{M} + \cos\bar{\theta}) - G\,(1 + 2\bar{M}\cos\bar{\theta} + \bar{M}^2)\}, \tag{6}$$

$$G = l\,\frac{1 - ([v]/v)\sin^2\psi}{1 - 2l\,([v]/v)\sin^2\psi},\quad l = \frac{c_p}{\beta\,[p]}, \tag{A}$$

c_p and β are the heat capacity and the coefficient of thermal expansion at constant pressure. For normal incidence we obtain

$$\delta p\,(1 + M)\,\{1 - l\,(1 + M)\} = \delta\bar{p}\,(\bar{M} + 1)\,\{\bar{M} - l\,(\bar{M} + 1)\}. \tag{7}$$

For an ideal gas (7) reduces to the formulas given in Refs. 1 and 5.

It is planned to describe the work in greater detail in the Acoustic Journal.

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*We follow the notation of Refs. 4 and 5.

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[5] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media), Gostekhizdat (1953) p. 409.

Translated by G. Volkoff
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INDEX OF THE ENERGY SPECTRUM OF THE PENETRATING COMPONENT IN EXTENSIVE AIR SHOWERS WITH A SPECIFIED NUMBER OF PARTICLES

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THE energy spectrum of the penetrating component was studied in Tbilisi at 400 m elevation at the depths of 61, 127, and 162 meters water equivalent (m w.e.).

An array consisting of three Geiger-Müller counters, each with an area of 0.0257 m^2, placed on the earth surface in a circle with a radius of ~ 1 m, was used for shower selection. The surface part of the apparatus consisted further of four correlated hodoscopes, each with a total sensitive area of 0.3 m^2. One of the hodoscopes was placed under the selecting system, two were placed on the two sides at a distance of 10 m, and the fourth at a distance of 20 m in the direction perpendicular to the line joining the first three hodoscopes. The underground part consisted of two layers of counters, separated by 8 cm of lead. Each layer consisted of 30 counters, the total area being about 1 m^2. Coincidences between the two layers were recorded. The master pulse, formed by a triple coincidence of the counters of the selecting system, operated a cine camera which photographed a panel of neon bulbs connected to the underground array and all hodoscopes. The selecting system was also displayed on the panel. The shower size was determined graphically, using curves drawn according to the Nishimura-Kamata distribution for $s = 1.2$ (Ref. 1). The same method was used to determine the distance between the shower axis and the center of the selecting system.

The mean number of particles in the measured group of showers was equal to 2.9×10^5. The number of particles of each individual shower was determined with an accuracy of ± 50%.

The mean density of penetrating particles ρ_p was found using the formula

$$\rho_p = \frac{1}{s_1} \ln \frac{N}{N - N_p}, \tag{1}$$

where s_1 is the area of the underground group of counters, N is the number of recorded showers, and N_p is the number of counts of the underground array.

The obtained data are given in Table I.

A method of data reduction analogous to that used by Sakvarelidze[2] yields a slightly divergent spectrum with exponent larger than that obtained in Ref. 2. It should be taken into account that, in our estimate, the mean shower size detected in the experiment of Sakvarelidze was one order of magnitude smaller and that our selecting system favored showers falling within a smaller radius. The results of the present experiment and that of Sakvarelidze, although not in contradiction, cannot therefore be compared quantitatively.

TABLE I

Depth (m w.e.)	Number of counts of the selecting system	Number of counts of the underground array	Density of penetrating particles (m^{-2})	Effective distance (m)
61	262	101	0.49 ± 0.07	16
127	519	93	0.20 ± 0.02	27
162	856	115	0.15 ± 0.02	29

Table I contains also the effective distances between the shower axis and the penetrating particle detector which was placed at different depths. It can be seen from the table that the effective distance increased slightly with the depth, which is due to the zenith-angle distribution of extensive air showers.

The following expression for the zenith-angle distribution, given by Greisen[1], was used in calculation of the effective distance:

$$J_\vartheta = J_0 \cos^{8.3} \vartheta. \tag{2}$$

The density of penetrating particles measured by us at the depth of 61 m w.e. below the selecting system was found to be equal to $(0.49 \pm 0.07)\ m^{-2}$. Within experimental errors, this in in agreement with the value $(0.55 \pm 0.06)\,m^{-2}$ obtained for the density of penetrating particles of a similar group of showers at the same depth, using the same relative position of the detecting system and the penetrating particle detector.[3] This makes it possible to use the results on lateral distribution obtained in Ref. 3:

$$\rho_p = (0.66\pm0.09)\exp[-(0.00058\pm0.00009)\,r^2]. \tag{3}$$

In consequence, a density of $(0.42 \pm 0.06)\ \mathrm{m}^{-2}$ is found for the effective distance of 28 m.

The results for all three depths are then found to be consistent. These data, expressed in energy units which account for the atmospheric depth and refer to approximately equal effective distances, are given in Table II.

TABLE II

Depth (Bev)	Effective Distance (m)	Density of Penetrating Particles (m^{-2})
14.9	28	0.42±0.06
29.5	27	0.20±0.02
37.1	29	0.15±0.02

The energy spectrum of penetrating particles of an extensive air shower calculated from the above data for a mean distance of 28 m from the shower axis can be represented by a power law $\rho_p = AE^{-\gamma}$ with $\gamma = 1.09 \pm 0.21$.

In conclusion it should be noted that a tendency has been observed of a variation of the spectrum exponent with decreasing energy of the shower. Further experiments are being carried out to investigate this effect and to improve the degree of knowledge of the shape of the spectrum.

The authors wish to express their gratitude to M. F. Bibilashvili for his help in carrying out the experiment and active participation in the discussion of results, and to R. B. Liudvigov and Sh. P. Abramidze of the Physics Institute of the Academy of Sciences of the Georgian S.S.R., and student G. G. Managadze of the Tbilisi State University for participation in the experimental work and the reduction of data.

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TRANSITION EFFECT OF STARS IN A LEAD ABSORBER

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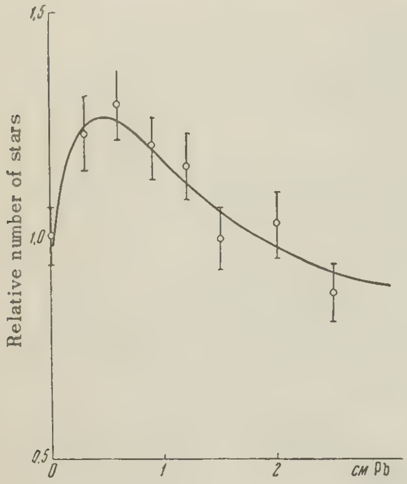
EXPERIMENTS carried out at mountain altitudes[1-3] and in the stratosphere[4,5] have established the existence of a transition effect of stars by means of photographic emulsions. However, such effect was found in other similar experiments.[6,7] Interest in this phenomenon was increased after a transition effect for stars was also found in a graphite absorber,[8] and in this way it was shown that the explanation of the transition effect in lead by means of the photon component of cosmic rays was not tenable.

In this note we present results of a control experiment undertaken in order to observe by means of photographic emulsions the transition effect for stars in lead absorbers.

Flat lead absorbers were placed one above another. Each layer of lead had an area of $40 \times 60\ \mathrm{cm}^2$. The photographic films were placed horizontally singly at each depth of the lead absorber. The exposure was carried out at an altitude of 3100 m above sea level. The figure shows the results of the experiment. As may be seen from the graph the maximum of the transition effect amounts to 30%.

Since the transition effect is not found by all observers, it should be noted that at mountain altitudes its magnitude is not great (according to the data of various authors it amounts to 15 – 30%); at the same time the average efficiency of a scanner is 80%, therefore the results are strongly affected by the method and the quality of scanning. Thus, for example, as a result of carrying out a single scanning the existence of the maximum in the curve is not very pronounced. We have employed three fold scanning of the same volume of photographic emulsion in which three different scanners took part. This has greatly increased the efficiency of scanning and has lead to the discovery of a sharply pronounced maximum. The small depth at which the maximum in the transition effect of stars occurs may also explain the absence of a transition effect under unfavorable experimental conditions — unfavorable absorber geometry, thick stacks of photographic emulsion, nearby massive objects, etc.



The results obtained by us do not disagree with the results of other investigators.

The authors express their gratitude to Prof. E. L. Andronikashvili who provided the facilities for carrying out this work, and to L. A. Razdol'skaia, E. I. Kierkesali, K. V. Mandritskaia, and M. M. Freidlin for aid in scanning the photographic emulsion layers.

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ELECTROMAGNETIC INTERACTION WITH PARITY VIOLATION

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UNTIL the discovery of violation of parity it was assumed that the interaction of an elementary particle of spin $\frac{1}{2}$ with a weak electromagnetic field was completely described by three terms in the energy

$$q\varphi,\ \mu(\sigma\mathbf{H}),\ a\,\mathrm{div}\,\mathbf{E} = 4\pi a\rho,$$

where σ is the spin, q the charge, μ the magnetic moment and the constant a characterizes the field of a spherical capacitor, equal to zero outside but interacting with a charge ρ inside.[1]

In their well known article about nonconversation of parity, Lee and Yang[2] indicate the possibility of an electrical dipole moment, i.e., an interaction $d(\boldsymbol{\sigma}\cdot\mathbf{E})$. However, if, with parity violation, there is conservation of invariance with respect to combined inversion (and consequently, also with respect to reflection in time) then, as Landau[3] has shown, no dipole moment is possible. It is easiest to see this by noting that under time inversion σ changes sign but **E** does not.

Which electromagnetic interactions, forbidden in the case of parity conservation, are possible in the theory of combined inversion?

The interaction $(\sigma \cdot A)$ is not allowable because of gauge invariance. There remains

$$b(\sigma \Delta A) = b(\sigma\, \mathrm{curl}\, H) = \frac{4\pi b}{c}(\sigma j).$$

Here $j = \rho v$ is the current density which produces the magnetic field H. Under conservation of parity such a term would be pseudoscalar (σ is a pseudovector, j a vector) and could not occur in the expression for the energy. On the other hand, both σ and j change sign under time reflection. The moment of force corresponding to such an interaction energy is $M = (4\pi b/c)[\sigma \times j]$.

Such an interaction is directly obtained from a model of virtual decay of a spin-$\frac{1}{2}$ particle A into a particle B of spin 0 and a particle C of spin $\frac{1}{2}$.[4] If this decay depends on a weak interaction which does not conserve parity, then two particles can simultaneously be produced in either S- or in P-states. Invariance of the theory relative to combined inversion corresponds to the relation of phases of the S- and P-waves which is such that the probabilities of finding the particle C above and below the equatorial plane (perpendicular to the direction of spin of the particle A) are equal, so that there is no electrical dipole moment. However, the virtual particles C in the equatorial plane have a transverse polarization; their spin σ_C has a component directed along $[r \times \sigma_A]$. Thus, around the spin axis of A there is a ring of elementary magnets — virtual particles C with spin along the equator. The magnetic interaction, proportional to $(\sigma_C \cdot H)$ gives a term proportional to the integral $\oint H\, d\ell$, taken along the equator. Such an integral can be expressed in terms of curl H. It is relevant to remember Ioffe's work:[5] for a real decay, combined inversion gives an asymmetry in the direction of emission and a longitudinal polarization, whereas the second variant of Ioffe (invariance relative to charge conjugation) gives a symmetrical emission and transverse polarization of the emitted particles. In virtual decays, the correlation is the other way around; as noted by Ioffe, in the latter variant there can be a dipole moment, whereas in combined inversion there is no dipole moment, but, as can be seen from Ref. 4 and this note, there is a transverse polarization leading to the interaction $(\sigma \cdot \mathrm{curl}\, H)$. From the point of view of classification of magnetic properties of the particle, the interaction $(\sigma \cdot \mathrm{curl}\, H)$, obviously, does not correspond to any magnetic multipole (dipole, quadrupole, etc.); we will call it an "anapole." For an understanding of the anapole, transformation of the energy to the form $(\sigma \cdot j)$ is essential; the anapole interacts only with the current which flows into the point at which the particle is to be found. Consequently, the external field of the anapole is identically zero (more accurately — falls off just as the probability density in the cloud of virtual particles C falls off).

In this connection, the anapole is analogous to a spherical capacitor in which the field differs from zero only inside the capacitor. The difference from a capacitor comes from the fact that the anapole is a vector, has a definite direction (along the spin of the particle A considered), whereas a spherical capacitor is characterized by a scalar quantity.

A classical model of the anapole can be represented as a wire helix (solenoid) bent in a ring (toroid). The current flowing through the helix creates a magnetic field only inside the toroid. If the toroid is rigid, then no magnetic field produced by external currents can act on the toroid as a whole. However, if this wire toroid is immersed in a solution of electrolyte which fills also the space inside the solenoid and a current is passed through the electrolyte, then a moment of force, proportional to the sine of the angle between the toroid and the direction of the current in the electrolyte, will act on the toroid. This corresponds to an interaction energy $(\sigma \cdot j)$, since the axis of the toroid is directed along σ.

The anapole moment of elementary particles, i.e., the constant b, can be estimated by multiplying the magnetic moment by the Compton wavelength $\hbar/mc$ and by the square if the dimensionless constant of weak interaction f^2, i.e., is of order $10^{-26}\mu$, in so far as only the interaction in which parity is not conserved gives the anapole. It is possible that, for experimental observation, the interaction of the anapole, not with the current but with a varying electric field giving a moment of force $M = (4\pi b/c)[\sigma \times \dot{E}]$, is more important. However, in view of the smallness of b, at the present time such an experiment would not seem to be possible.

The anapole interaction is an example which directly refutes the assertion of V. G. Solov'ev,[6] according to which combined inversion and gauge invariance lead to conservation of spatial parity in electrodynamics.

After finishing this work, I learned that V. G. Vaks obtained independently results analogous to ours.

I should like to use this opportunity to thank B. L. Ioffe, A. S. Kompaneets (who proposed the name "anapole"), L. D. Landau, and Ia. A. Smorodinski for discusions.

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SPACE-ENERGY DISTRIBUTION OF NEUTRONS IN A HEAVY GASEOUS MODERATOR

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THE theory of neutron thermalization in a heavy (atomic weight $M \gg 1$) monoatomic gas with constant mean free path λ and constant neutron lifetime τ has been discussed in a number of papers.[1-3] However, the majority of the results refers to the energy distribution only. The space-energy distribution function has been found only in the region of relatively large energies.[2] In the case of weak absorption this problem can be solved exactly.

The equation for the space-energy distribution function[2] may be written as follows:

$$-\alpha\psi(\mathbf{r},x)+\beta x\nabla^2\psi(\mathbf{r},x)+(3-2x^2)\frac{\partial\psi(\mathbf{r},x)}{\partial x}+x\frac{\partial^2\psi(\mathbf{r},x)}{\partial x^2}=0,\qquad \alpha=(2M\lambda/\tau)\sqrt{m/2kT},\quad \beta=2M\lambda^2/3. \tag{1}$$

where x^2 is the neutron energy in units of kT (T — temperature of the moderator), $\psi(\mathbf{r}, x)$ is the space-energy distribution function divided by $x^2e^{-x^2}$, and m is the neutron mass.

For a moderator of finite dimensions one may obtain a solution of Eq. (1) in the form of an expansion in a complete set of orthonormal functions $R_l(\mathbf{r})$ of the Laplacian for the corresponding boundary value problem $[\nabla^2R_l(\mathbf{r}) + \Omega_lR_l(\mathbf{r}) = 0]$, i.e.,

$$\psi(\mathbf{r},x)=\sum_l R_l(\mathbf{r})\,n_l(x). \tag{2}$$

then each of the functions $n_l(x)$ should satisfy the equation

$$x\,d^2n_l/dx^2+(3-2x^2)\,dn_l/dx-(\alpha+\beta x\Omega_l)\,n_l=0. \tag{3}$$

Making use of the requirement that $n_l(x)$ be finite as $x \to 0$, this equation may be transformed into an integral equation of the Volterra type

$$\begin{gathered} n_l(x)=C_l\Phi(a,2,x^2)+\alpha\int_0^x n_l(t)K(x,t)\,dt,\\ K(x,t)=\frac{1}{2}\Gamma(a)\,t^2e^{-t^2}[\Psi(a,2,t^2)\,\Phi(a,2,x^2)-\Phi(a,2,t^2)\,\Psi(a,2,x^2)], \end{gathered} \tag{4}$$

the solution of which, as is well known, is of the form

$$n_l(x)=C_l\sum_{m=0}^{\infty}\alpha^m\varphi_m(x),\qquad \varphi_0(x)=\Phi(a,2,x^2);\quad \varphi_{m+1}(x)=\int_0^x\varphi_m(t)\,K(x,t)\,dt. \tag{5}$$

Here $\Phi(a, b, z)$ is confluent hypergeometric function and

$$\Psi(a,b,z)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-zt}t^{a-1}(1+t)^{b-a-1}dt$$

is the other linearly independent solution of the hypergeometric equation; $a = \beta\Omega_{\ell}/4$.

The normalization constants C_{ℓ} may be obtained by comparing the assymptotic form of $n_{\ell}(x)$ for large x with the results of a calculation in the Fermi Age approximation. Thus, neglecting absorption for the sake of simplicity, one has for the case of a unit intensity source of neutrons at the point $\mathbf{r} = \mathbf{r}_0$ in the age approximation, the following

$$\psi_{\text{Age}}(\mathbf{r}, x) = \frac{\lambda M}{2}\sqrt{\frac{m}{2kT}}\,\frac{x^2}{x^4}\sum_{l} R_l(\mathbf{r})\left(\frac{x}{x_0}\right)^{\beta\Omega_l/2}, \tag{6}$$

where x_0 is the source neutrons speed. Hence

$$\psi(\mathbf{r}, x) = \frac{\lambda M}{2}\sqrt{\frac{m}{2kT}}\sum_{l} R_l(\mathbf{r}_0)\,R_l(\mathbf{r})\,x_0^{-\beta\Omega_l/2}\Gamma\left(\frac{\beta\Omega_l}{4}\right)\Phi\left(\frac{\beta\Omega_l}{4}, 2, x^2\right) \tag{7}$$

In the case of a source located in an infinite homogeneous medium the sum over ℓ must be replaced by the corresponding integral.

A detailed discussion of applications of the above results to various special cases will be published later.

In conclusion I express deep gratitude to F. L. Shapiro for valuable discussions in the process of this work.

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INFLUENCE OF FINITE NUCLEAR SIZE ON EFFECTS CONNECTED WITH PARITY NONCONSERVATION IN BETA DECAY

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FOR β-decay, particularly for forbidden transitions, the action of the nuclear field is of importance. As in Ref. 1, we take for the electron wave function

$$\psi_e = \begin{pmatrix}\varphi_e\\ \chi_e\end{pmatrix}, \quad \begin{aligned}\varphi_e &= [\alpha_0 + ipr\alpha_1 + i(\sigma\mathbf{r})(\sigma\mathbf{n})\beta_c]\,u_\xi,\\ \chi_e &= [\beta_0 + ipr\beta_1 + i(\sigma\mathbf{r})(\sigma\mathbf{n})\alpha_c](\sigma\mathbf{n})\,u_\xi,\end{aligned} \tag{1}$$

where

$$\mathbf{n} = \frac{\mathbf{p}}{p}, \quad \alpha_0 = \sqrt{\frac{\pi}{2p\varepsilon}}\,ie^{-i\delta_{-1}}g_{-1}, \quad \beta_0 = \sqrt{\frac{\pi}{2p\varepsilon}}\,f_1e^{-i\delta_1}, \quad \alpha_1 = \sqrt{\frac{\pi}{2p\varepsilon}}\,\frac{3}{pr}\,e^{-i\delta_{-2}}g_{-2}, \quad \beta_1 = \sqrt{\frac{\pi}{2p\varepsilon}}\,\frac{3}{ipr}\,e^{-i\delta_2}f_2,$$

$$\beta_c = \sqrt{\frac{\pi}{2p\varepsilon}}\,\frac{1}{r}\left(e^{-i\delta_1}g_1 - e^{-i\delta_{-2}}g_{-2}\right), \quad \alpha_c = \sqrt{\frac{\pi}{2p\varepsilon}}\,\frac{1}{ir}\left(e^{-i\delta_{-1}}f_{-1} - e^{-i\delta_2}f_2\right)$$

($m_e = c = \hbar = 1$), g_K, f_K are the inside-the-nucleus solutions of the radial Dirac equation joined with the outside solutions at $r = r_0$; δ_K is the phase.[2] We write the β-interaction Hamiltonian as follows

$$H = \sum\left\{g_i(\bar{\psi}_2O_i\psi_1)\left(\bar{\psi}_eO_i\,\frac{1-\gamma_5}{2}\,\psi_\nu\right) + g_i'(\bar{\psi}_2O_i\psi_1)\left(\bar{\psi}_eO_i\,\frac{1+\gamma_5}{2}\,\psi_\nu\right)\right\} \tag{2}$$

(summation over $i = S, T, V, A, P$). If the two-component neutrino theory[3-5] holds, then emission of an antineutrino together with an electron corresponds to $g_i' = 0$, whereas emission of a neutrino corresponds to $g_i = 0$.

The results for allowed and first forbidden β-transitions are expressed in terms of the known tabulated functions L_0, M_0, N_0, P_0, Q_0, R_0, L_1, P_1.[6–8]

Taking finite nuclear size into acoount has in practice no effect on allowed and unique $\Delta j = 2$ (yes) transitions since these depend on L_0, P_0 and L_0, P_0, L_1, P_1 respectively, for which finite nuclear size is unimportant.[9] For the Coulomb transitions M_0 and Q_0 are relevant; these are significantly affected by finiteness of nuclear size, but since they are both multiplied by the same factor they will give no effect in practice. Thus nuclear size will affect only $0 \rightarrow 0$ (yes) transitions provided the pseudoscalar covariant is present in β-decay.

If one assumes that the axial covariant is absent, then the longitudinal polarization of electrons and the electron-neutrino angular correlation are given by

$$\langle \sigma n \rangle = (A_{av} - A_{v})/(C_{av} + C_{v}), \cdot\ W_{ev}(\theta) = 1 + \langle \sigma n \rangle \cos\theta, \tag{3}$$

where

$$A_{av} = -|g_T \int \sigma r|^2 \{ \tfrac{1}{9} q^2 \sqrt{L_0^2 - P_0^2} + \sqrt{M_0^2 - Q_0^2} - \tfrac{1}{3} q (\sqrt{(L_0 + P_0)(M_0 + Q_0)} + \sqrt{(L_0 - P_0)(M_0 - Q_0)})$$

$$- (\sqrt{(L_0 + P_0)(M_0 + Q_0)} + \sqrt{(L_0 - P_0)(M_0 - Q_0)}) - \tfrac{2}{3} q \sqrt{(L_0^2 - P_0^2)} \operatorname{Re} \lambda_P + (\sqrt{(L_0 + P_0)(M_0 + Q_0)}$$

$$- \sqrt{(L_0 - P_0)(M_0 - Q_0)}) \cot(\delta_{-1} - \delta_1) \operatorname{Im} \lambda_P + \sqrt{L_0^2 - P_0^2} |\lambda_P|^2 \} \sin(\delta_{-1} - \delta_1); \tag{4}$$

$$C_{av} = \{ \tfrac{1}{9} L_0 q^2 + M_0 + \tfrac{2}{3} q N_0 + 2 \left(N_0 + \tfrac{1}{3} L_0 q \right) \operatorname{Re} \lambda_P + L_0 |\lambda_P|^2 \} |g_T \int \sigma r|^2 \quad \lambda_P = -i g_P \int \gamma_5 / g_T \int \sigma r.$$

A_ν and C_ν follow from (4) upon replacement of g_P by g'_P and g_T by g'_T.

For $Ze^2 \ll 1$, finite nuclear size effects are unimportant and all quantities can be written explicitly.[6] Since expressions for Q_0 and P_0 as well as for $\sin(\delta_{-1} - \delta_1)$ are not available, we give them here:

$$P_0 = 1/\varepsilon, \ Q_0 = pv/9 \cdot - Z^2 e^4 / 4\varepsilon r_0^2, \ \sin(\delta_{-1} - \delta_1) = (1 + Z^2 e^4 p^{-2})^{-1/2}. \tag{5}$$

If the pseudoscalar covariant is absent, the two-component neutrino theory holds, and λ_P is real, then the polarization differs insignificantly (~ 3%) from v/c. In the presence of the pseudoscalar covariant the polarization can differ significantly from v/c only in a relatively narrow range of values of λ_P. For example, for Pr^{144} the polarization differs from v/c for $\lambda_P \sim 10 \pm 1$ and may even change sign. If the two-component neutrino theory does not hold then the polarization obviously can take on any value.

In conclusion I wish to thank V. B. Berestetskii, B. L. Ioffe and K. A. Ter-Martirosian for their suggestion of this problem and discussions.

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ONE POSSIBILITY OF BARYON CLASSIFICATION

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FROM the baryon wave equation[1] one finds as the general expression for the baryon's electric charge*

$$q = -\varepsilon(t_3 + v_3). \tag{1}$$

Here t_3 stands for the eigenvalue of the operator T_3, and v_3 is a certain number which changes in going from one multiplet to another. In our treatment v_3 replaces the Gell-Mann number; it could be called the eigenvalue of the third component of the isotopic momentum operator V_3.

The requirement that all elementary particles should have charge $+\epsilon$, $-\epsilon$ or 0 leads to the following rules:

(a) t_3 and v_3 are simultaneously integral or half-integral.

(b) $|t_3 + v_3|$ may not exceed unity. From these rules one obtains quantum numbers determining the baryon type (see table). For the Λ^0 particle the values $v_3 = \pm 1$ are also possible; however these should apparently be excluded.

Baryon Type	t_3	v_3
P	$-1/2$	$\left.\right\}-1/2$
n	$+1/2$	
Λ^0	0	0
Ξ^0	$-1/2$	$\left.\right\}+1/2$
	$+1/2$	
Σ^-	1	$\left.\right\}\ 0$
Σ^0	0	
Σ^+	-1	

The values $v_3 = \pm\frac{1}{2}$ indicate that the nucleon and Ξ-hyperon doublets are components of a doublet in the v_3-space. Indeed, it is easy to write a general equation for the nucleon and Ξ-hyperon:

$$[\Gamma^\nu(\partial/\partial x_\nu) - k_0 I \exp(aT_3 + bV_3)]\psi = 0, \tag{2}$$

where Γ^ν, I, T_k and V_k are 16 × 16 matrices

$$\Gamma^\nu = 1^{\text{IV}} \times \gamma^\nu, \quad V_k = (1^{\text{II}} \times \sigma_k) \times \gamma^5,$$
$$T_k = (\sigma_k \times 1^{\text{II}}) \times \gamma^5, \quad I = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \times 1^{\text{IV}}. \tag{3}$$

For the constants a, b, $m_0 = k_0\hbar/c$ the following values should be taken:

$$m_0 = 2181\, m_e, \quad a = 3\alpha/10\pi, \quad b = 0.1714.$$

It is interesting to note that the value of m_0 coincides with the mass of the Λ^0 particle. This makes it possible to describe the Λ^0 particle as a nucleon in the special singlet state $t_3 = 0$, $v_3 = 0$.

The constant a, characteristic of mass differences within a t_3 multiplet, is of the order of magnitude α/π. The constant b, characteristic of mass differences between nucleon, Ξ hyperons and the Λ^0 particle, is of the order of magnitude g/π, where g is the strong coupling constant. This shows that mass differences within a multiplet are caused by electro-magnetic interactions, whereas mass differences between nucleons, Ξ hyperons, Σ hyperons and the Λ^0 particle are due to strong interactions.

It follows from the above considerations that nucleons, Ξ hyperons, and the Λ^0 particle are closely related to each other, whereas the Σ hyperons form a separate group. One may suppose that besides nucleons, Ξ hyperons also are present inside nuclei (the existence of the Λ^0 particle in nuclei has been proven).

In conclusion I consider it my pleasant duty to express sincere gratitude to P. Kard for criticism of this work.

Note added in proof (November 21, 1957). The main results, following from our equations, are in full agreement with the Peaslee scheme with which the author only became familiar after the present work was submitted to press.

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*This result coincides with the expression for the charge given by Salam and others.[2]

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ERRATA TO VOLUME 6

Page	Line	Reads	Should Read
643	16 from bottom	where $\kappa = \pi a^2\Omega - \ldots$	where $\kappa = \pi a^2\Omega\varphi - \ldots$
690	8 from bottom	$\ldots \sin [- \ldots$	$\ldots \sin \vartheta [- \ldots$
	5 from bottom	$\ldots \sin 2\vartheta \sqrt{\frac{1}{3}} \ldots$	$\ldots \sin 2\vartheta \left[\sqrt{\frac{1}{3}} \ldots\right.$
809	9 from top	$\ldots\left(\frac{1}{2 \sinh u} + \ldots\right.$	$\ldots\left(\frac{1}{\sinh u} + \ldots\right.$
973	unnumbered equation	$\ldots C_{n\mu-\mu'}^{S'-\mu';\, S\mu} T_{\mu'-\mu'}^{(n)}.$	$\ldots C_{n\mu-\mu'}^{S'-\mu';\, S\mu} \langle S' \Vert T^{(n)} \Vert S^{-1} \rangle$ $\times T_{\mu'-\mu}^{(n)}.$
975	5 from bottom	...of a particle by a nucleus ...	...of a particle in state a by a nucleus ...
992	Eq. (18)	$\ldots \tau_1\tau_2^{-2}/2\hbar^1 \ldots$	$\ldots \tau_1\tau_2^{-1}/2\hbar^2 \ldots$

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